Introduction to algebraic topology : Assignment 1

Exercise 1. (i) Let $\mathbb{D}^2$ be the closed unit disk in $\mathbb{R}^2$. Prove that every continuous self map $f : \mathbb{D}^2 \longrightarrow \mathbb{D}^2$ has a fixed point. This is Brouwer's fixed point in dimension 2.

(ii) Prove that every 3×3 real matrix whose entries are all positive admits a positive eigenvalue.

Hint : Let A be such a matrix. Consider the subset $\mathrm{P} \subset \mathbb{S}^2$ of unit vectors in $\mathbb{R}^3$ whose coordinates are all non-negative. Show that A induces a map $f : P \longrightarrow P$ defined by $f(x) = \frac{A(x)}{||A(x)||}$ and find a fixed point of f.

Exercise 2. Let M be the Möbius band, defined as the quotient of the square $[0,1] \times [0,1]$ by the equivalence relation $(0,t) \sim (1, 1-t)$ endowed with the quotient topology.

(i) Compute the fundamental group of M at a point.

Hint : Let D be the diagonal of the square $[0,1] \times [0,1]$ and consider the map $[0,1] \times [0,1] \longrightarrow D$ mapping (x,y) to (x,x). Show that this is a deformation retraction. How does this help ?

(ii) Define the Klein bottle K as the quotient of the square $[0,1] \times [0,1]$ by the equivalence relation $(0,t) \sim (1,1-t)$ and $(s,0) \sim (s,1)$. Compute the fundamental group of K at a point.

Hint : You may want to use the Seifert-Van Kampen theorem and the result of 1).

Exercise 3. Let M and N be n-dimensional topological manifolds. Let $i_1 : \mathbb{B}^n \longrightarrow M$ and $i_2 : \mathbb{B}^n \longrightarrow N$ be embeddings (ie homeomorphisms onto their images) of the open unit ball of $\mathbb{R}^n$ into M and N. Set $\mathbb{B}' = \frac{1}{2}\mathbb{B}$.

Define the connected sum of M and N to be the space $M \sharp N := \frac{(M \setminus i_1(\mathbb{B}')) \coprod (N \setminus i_2(\mathbb{B}'))}{\mathcal{R}}$ where $\mathcal{R}$ is the relation that identifies $i_1(x)$ and $i_2(x)$ for $x \in \overline{\mathbb{B}'} \setminus \overset{\circ}{\mathbb{B}'} \simeq \mathbb{S}^{n-1}$.

(i) If $n > 2$, prove that $\pi_1(M \sharp N, p) = \pi_1(M, q) \star \pi_1(N, r)$, where p is any point of the connected sum and $(q, r) \in (M \setminus i_1(\mathbb{B}')) \coprod (N \setminus i_2(\mathbb{B}'))$ maps to p.

(ii) Compute the fundamental group of a k-times iterated connected sum of $\mathbb{S}^1 \times \mathbb{S}^4$.

Please read pages 255 to 257 of Lee's book "Introduction to topological manifolds" (Wedge Sums) and compare 3.i) to the statement and proof of theorem 10.7.