Introduction to algebraic topology : Assignment 2

Exercise 1. *Recall that a topological group is a group G endowed with a topology such that the multiplication and inverse maps are continuous.*

(i) Let G be a connected and locally path connected topological group with neutral element e and $p : E \longrightarrow G$ be a covering map with E connected. Show that E can be made into a topological group such that p is a group homomorphism.

(ii) If $p : G' \longrightarrow G$ is a covering of topological groups, show that $\ker(p)$ is a discrete subgroup of the center $Z(G')$ of G'.

(iii) Let G be a topological group and $\Gamma \subset Z(G)$ a discrete subgroup of its center. Show that the quotient topology makes G/Γ into a topological group and that the quotient map $\pi : G \longrightarrow G/\Gamma$ is a covering. Compute its automorphism group.

(iv) Suppose furthermore that G has a universal covering $u : \tilde{G} \longrightarrow G$. Show that $\ker(u)$ and $\pi_1(G, e)$ are isomorphic as groups.

Exercise 2. For a positive integer n, let C_n be the circle of radius $\frac{1}{2^n}$ centered at the point $(\frac{1}{2^n}, 0)$. Let C be the union of all these circles (it is sometimes called a clamshell or Hawaiian earring).

(a) Show that C is connected and locally path-connected, but not semilocally simply connected and therefore has no universal covering space.

(b) Identify C with $C \times \{0\} \subset \mathbb{R}^3$ and let X be the cone over C, that is the union of all segments from points in C to the point $(0, 0, 1)$.
Show that X is semilocally simply connected but not locally simply connected.

Exercise 3. Let $\mathcal{E}$ be the *figure-eight* space, union of the two circles of radius 1 and respective centers $(0, 1)$ and $(0, -1)$. Let $X \subset \mathbb{R}^2$ be the union of the x-axis with infinitely many unit circles centered at $\{2\pi k + i,\ k \in \mathbb{Z}\}$. Let $p : X \longrightarrow \mathcal{E}$ be the map sending each circle onto the upper circle of $\mathcal{E}$ by translating along the x-axis and that sends the x-axis onto the lower circle by $x \mapsto ie^{ix} - i$.

(i) Show that p is a covering map.

(ii) Compute the subgroup $p_*\pi_1(X, 0) \subset \pi_1(\mathcal{E}, 0)$ (Give an expression of it in terms of the generators of $\pi_1(\mathcal{E}, 0)$).

(iii) Determine the automorphism group $\mathrm{Aut}(X/\mathcal{E})$ of p.

(iv) Is p a normal covering ?