Introduction to algebraic topology : Assignment 3

Exercise 1. If Γ is a group and $\alpha, \beta \in \Gamma$, define the commutator of α and β to be $[\alpha, \beta] = \alpha^{-1}\beta^{-1}\alpha\beta$. The commutator subgroup (also called the derived subgroup) of Γ is the subgroup generated by all the commutators of elements of Γ. It is denoted by $[\Gamma, \Gamma]$.

(a) Show that the commutator subgroup of any group is normal.

If Γ is a group we define $\Gamma_{ab} = \Gamma/[\Gamma, \Gamma]$

(b) What universal property does Γ_{ab} have ? Prove your answer.

Let X be a connected, locally path connected space that has a universal covering $\tilde{X}$. Fix $x \in X$. Define $\tilde{X}_{ab} = \tilde{X}/[\pi_1(X, x), \pi_1(X, x)] \longrightarrow X$.

(c) Show that $\tilde{X}_{ab} \longrightarrow X$ is a Galois covering of group $\pi_1(X, x)_{ab}$. The latter is called the abelian fundamental group.

A covering is said to be abelian if it is Galois with abelian Galois group.

(d) Show that any connected abelian covering of X has the form $\tilde{X}_{ab}/H \longrightarrow X$, for some subgroup H of $\pi_1(X, x)_{ab}$.

The same kind of things can be said about field extensions. You can try to work it out for yourself : define an abelian field extension, the abelian Galois group of a field and prove a result similar to (d).

Exercise 2. *Base change.*

Let $f : (Z, z) \longrightarrow (X, x)$ be a map of connected, locally path connected spaces that have universal covering spaces.

1) Describe a natural functor $f^* : \mathrm{Cov(X)} \longrightarrow \mathrm{Cov(Z)}$ from coverings of X to coverings of Z. (Prove that your construction is well-defined and functorial)

2) Describe a natural functor $\mathrm{G} : \pi_1(X, x) - \text{sets} \longrightarrow \pi_1(Z, z) - \text{sets}$. (Prove that your construction is well-defined and functorial)

3) Show that these constructions commute with fiber functors, that is we have a commutative diagram of categories

$$\begin{array}{ccc} Cov(X) & \xrightarrow{f^*} & Cov(Z) \\ \downarrow{\scriptstyle Fib_x} & & \downarrow{\scriptstyle Fib_z} \\ \pi_1(X, x) - \text{sets} & \xrightarrow{G} & \pi_1(Z, z) - \text{sets} \end{array}$$

4) If $p : E \longrightarrow X$ is a covering of X, compute the automorphism group of the covering $f^*p : f^*E \longrightarrow Z$.

5) What is the covering induced by the homomorphism $f_* : \pi_1(Z, z) \longrightarrow \pi_1(X, x)$? (Prove your answer).

Exercise 3. Let X be the plane $\mathbb{R}^2$ with 2 points removed.

(i) What is the fundamental group of X ?

(ii) Show that there is a connected Galois cover $p\colon Y \to X$ with automorphism group isomorphic to S_3.