Introduction to algebraic topology : practice exercises

Exercise 1. If $\varphi : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is a map from the circle to itself, the map $\psi = \varphi/\varphi(1)$ maps 1 to 1 and hence induces a morphism $\pi_1(\mathbb{S}^1, 1) \simeq \mathbb{Z}$ to itself, which must be of the form $\gamma \mapsto \gamma^n$ for a uniquely determined integer n. We call this n the *degree* of the map φ.

We say that a map $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is *even* if $f(z) = f(-z)$ for all z and *odd* if $f(-z) = -f(z)$ for all z.

You may freely use the basic properties of the degree, especially 8.15, 8.16 and 8.17 from Lee's book.

1) Show that every even map has even degree.
2) Show that every odd map has odd degree, as follows.
   a) Let $p_2 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be the two-sheeted covering $z \mapsto z^2$. Show that if f is odd, there exists a map $g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ such that $p_2 \circ f = g \circ p_2$.
   b) Show that if $\deg(f)$ is even then g lifts to a map $\tilde{g} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ such that $p_2 \circ \tilde{g} = g$.
   c) Show that $\tilde{g} \circ p_2$ and f are both lifts of $g \circ p_2$ that agree at either 1 or -1. Derive a contradiction.
3) Show that for any continuous map $F : \mathbb{S}^2 \longrightarrow \mathbb{R}^2$ there exists a point $x \in \mathbb{S}^2$ such that $F(x) = F(-x)$.
   Hint : if not, $x \mapsto \frac{F(x)-F(-x)}{\|F(x)-F(-x)\|}$ *maps* $\mathbb{S}^2$ *to* $\mathbb{S}^1$ *and restricts to an odd map from* $\mathbb{S}^1$ *to itself.*

This is the Borsuk-Ulam theorem in dimension 2.

Exercise 2. Let X be the comb space, defined as the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0.1] \times [0, 1]$ together with the segments of the vertical lines $x = 1/2, 1/3, \ldots, 1/n, \ldots$ inside the square. Show that for any covering space $E \longrightarrow X$ there exists some neighbourhood of the left edge that lifts homeomorphically to E. Deduce that X has no simply connected covering.

Exercise 3. Let $p : E \longrightarrow X$ be a covering map. Show that E is compact if and only if X is compact and p is a finite-sheeted covering.