Introduction to algebraic topology : Week 1

Exercise 1. Let n be a fixed integer. Show that the Gram-Schmidt orthonormalization process provides a deformation retraction

$$r : \mathrm{GL}_n(\mathbb{R}) \longrightarrow \mathrm{O}_n(\mathbb{R})$$

from the set of invertible $n \times n$ real matrices to the set of $n \times n$ orthogonal matrices.

Exercise 2. Let X be a topological space. Suppose there exists a covering

$$X = \bigcup_{\lambda \in L} U_\lambda$$

of X by open subsets with the following properties :

(i) For any arbitrary $\lambda, \mu \in L$, there exists $\nu \in L$ such that $U_\lambda \cup U_\mu \subset U_\nu$.

(ii) For each $\lambda \in L$, U_λ is path connected and for every $x_\lambda \in U_\lambda$, the homomorphism

$$(i_\lambda)_* : \pi_1(U_\lambda, x_\lambda) \longrightarrow \pi_1(X, x_\lambda)$$

is null.

Prove that X is simply connected.

Exercise 3. Prove that every 3×3 real matrix whose entries are all positive admits a positive eigenvalue.

Hint : Let A be such a matrix. Consider the subset $\mathrm{P} \subset \mathbb{S}^2$ of unit vectors in $\mathbb{R}^3$ whose coordinates are all positive. Show that A induces a map $f : P \longrightarrow P$ defined by $f(x) = \frac{A(x)}{||A(x)||}$ and find a fixed point of f.

Exercise 4. Let (X, d_X) be a metric space. Fix a point $x_0 \in X$ and let $\mathcal{P}(X, x_0)$ be the set of paths

$$a : [0,1] \longrightarrow X$$

starting at x_0. Make $\mathcal{P}(X, x_0)$ into a metric space by setting $d(a,b) = \sup\{d_X(a(s), b(s)), s \in [0,1]\}$. Show that this metric space is contractible.