Introduction to algebraic topology : Week 12

Exercise 1. Let X be a topological space. Define the suspension SX of X by

$$SX = \frac{X \times [0,1]}{(x_1, 0) \sim (x_2, 0)\ ;\ (x_1, 1) \sim (x_2, 1)\ , \forall (x_1, x_2) \in X^2}$$

1) Show that suspension defines a functor Top $\longrightarrow$ Top (ie show that maps of topological spaces induce maps on suspensions).
2) Show that for any topological space X and any integer $p \geq 1$ we have $H_p(X) = H_{p+1}(SX)$.
3) Show that for all $n \geq 0$, $S\mathbb{S}^n = \mathbb{S}^{n+1}$.
4) Let $n \geq 1$. If $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ is map on the n-sphere ($n \geq 1$) by the preceding question the suspension Sf is a map on the $n+1$-sphere. Show that these maps have the same degree, ie

$$\deg(f) = \deg(Sf).$$

5) Let $1 \leq i \leq n$ and $f : \mathbb{S}^n \longrightarrow \mathbb{S}^n$ be the map sending $(x_1, ..., x_i, ..., x_{n+1})$ to $(x_1, ..., -x_i, ..., x_{n+1})$. Show that f has degree -1.
6) Show that the antipodal map

$$\begin{array}{cccc} a: & \mathbb{S}^n & \longrightarrow & \mathbb{S}^n \\ & x & \mapsto & -x \end{array}$$

has degree $(-1)^{n+1}$.

Exercise 2. In this exercise we prove that a sphere of positive even dimension cannot be given the structure of a topological group. Given a group G acting as a group of homeomorphisms of a space X, we say that G acts *freely* if the only element from G which has any fixed points is the identity element. Let g, h be two elements, unequal to the identity element, from a group G acting freely on $\mathbb{S}^n$, where $n > 0$ is even.
(a) Prove that both g and h have degree -1.
(b) Prove that gh is the identity element.
(c) Conclude that G is either $\mathbb{Z}/2\mathbb{Z}$ or the trivial group.
(d) Prove that $\mathbb{S}^n$ is not a topological group.