Introduction to algebraic topology : Week 2

Exercise 1. If M is a path connected manifold of dimension at least 3 , prove that removing a point from M has no effect on homotopy, ie prove that for every $p \in M$ and $x \neq p$,

$$\pi_1(M, x) = \pi_1(M \setminus \{p\}, x).$$

Reminder : a topological/differential manifold of dimension n is a Hausdorff topological space locally homeomorphic/diffeomorphic to $\mathbb{R}^n$

Exercise 2. Let $X \subset \mathbb{R}^3$ be the union of n lines through the origin. If $a \notin X$, compute $\pi_1(\mathbb{R}^3 \setminus X, a)$.

Exercise 3. If $x_1, ..., x_n$ are n distinct points in the plane, compute the fundamental group of $\mathbb{R}^2 \setminus \{x_1, ..., x_n\}$

Exercise 4. Let M and N be n-dimensional topological manifolds. Let $i_1 : \mathbb{B}^n \longrightarrow M$ and $i_2 : \mathbb{B}^n \longrightarrow N$ be embeddings (ie homeomorphisms onto their images) of the open unit ball of $\mathbb{R}^n$ into M and N. Set $\mathbb{B}' = \frac{1}{2}\mathbb{B}$.

Define the connected sum of M and N to be the space $M \sharp N := \frac{(M \setminus i_1(\overset{\circ}{\mathbb{B}'})) \coprod (N \setminus i_2(\overset{\circ}{\mathbb{B}'}))}{\mathcal{R}}$ where $\mathcal{R}$ is the relation that identifies $i_1(x)$ and $i_2(x)$ for $x \in \overline{\mathbb{B}'} \setminus \overset{\circ}{\mathbb{B}'} \simeq \mathbb{S}^{n-1}$.

(i) If $n > 2$, prove that $\pi_1(M \sharp N, p) = \pi_1(M, q) \star \pi_1(N, r)$, where p is any point of the connected sum and $(q, r) \in M \setminus i_1(\overset{\circ}{\mathbb{B}'}) \coprod (N \setminus i_2(\overset{\circ}{\mathbb{B}'}))$ maps to p.

(ii) Compute the fundamental group of a k-times iterated connected sum of $\mathbb{S}^1 \times \mathbb{S}^4$.