Introduction to algebraic topology : Week 3

Exercise 1. Show that every covering map is open (ie sends open sets to open sets).

Exercise 2. Let X_n be the union of n circles of radius 1 in $\mathbb{C}$ centered at the points $\{0, 2, 4, ..., 2n-2\}$. Observe that $\pi_1(X_n, 1)$ is a free group on n generators and describe loops representing the generators.

Let A, B, C be the first three circles and define the map $q : X_3 \longrightarrow X_2$ by

$$q(z) = \begin{cases} z & \text{if } z \in A \\ 2 - (z-2)^2 & \text{if } z \in B \\ 4 - z & \text{if } z \in C \end{cases}$$

Show that q is a covering map.

Exercise 3. Suppose that a group G acts on a space X in such a way that for all $x \in X$, there exists an open neighborhood V_x of x such that $V_x \cap gV_x = \emptyset$ for all $g \in G \setminus \{1\}$.

Show that the quotient map $X \longrightarrow X/G$ is a covering.

Deduce that if a finite group acts freely on a space X, the quotient map is a covering.

Exercise 4. Show that if a path-connected, locally path connected X has a finite fundamental group, every map $X \longrightarrow \mathbb{S}^1$ is nullhomotopic (ie homotopic to a constant map).

Exercise 5. Let X be the subspace of $\mathbb{R}^2$ consisting of the four sides of the square $[0,1] \times [0,1]$ together with the segments of the vertical lines $x = 1/2, 1/3, 1/4, ..., 1/n, ...$ inside the square. Show that for every covering $\tilde{X} \longrightarrow X$ there exists some neighborhood of the left edge of X which lifts homeomorphically to $\tilde{X}$. Deduce that X has no simply connected covering.