Introduction to algebraic topology : Week 4

Exercise 1. For $n \geq 1$, let $\mathbb{P}^n$ be the real projective n-space and $\mathbb{S}^n$ be the unit sphere in $\mathbb{R}^{n+1}$.

Define a map $p : \mathbb{S}^n \longrightarrow \mathbb{P}^n$ by sending x to the line going through the origin and x.

Show that p is a covering map and describe its fibers.

Exercise 2. Let $p : E \longrightarrow X$ be a covering map. Show that if X is connected, all fibers $p^{-1}(x)$ for $x \in X$ have the same cardinality.

Exercise 3. Let $p : E \longrightarrow X$ be a covering map. Show that E is compact if and only if X is compact and all the fibers $p^{-1}(x)$ for $x \in X$ are finite.

Exercise 4. If φ is a map $\varphi : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$, the map $\psi = \varphi/\varphi(1)$ maps 1 to 1 and hence induces a morphism $\pi_1(\mathbb{S}^1, 1) \simeq \mathbb{Z}$ to itself, which must be of the form $\gamma \mapsto \gamma^n$ for a uniquely determined integer n. We call this n the *degree* of the map φ.

We say that a map $f : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ is *even* if $f(z) = f(-z)$ for all z and *odd* if $f(-z) = -f(z)$ for all z.

You may freely use the basic properties of the degree, especially 8.15, 8.16 and 8.17 from Lee's book.

1) Show that every even map has even degree.
2) Show that every odd map has odd degree, as follows.
   a) Let $p_2 : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ be the two-sheeted covering $z \mapsto z^2$. Show that if f is odd, there exists a map $g : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ such that $p_2 \circ f = g \circ p_2$.
   b) Show that if $\deg(f)$ is even then g lifts to a map $\tilde{g} : \mathbb{S}^1 \longrightarrow \mathbb{S}^1$ such that $p_2 \circ \tilde{g} = g$.
   c) Show that $\tilde{g} \circ p_2$ and f are both lifts of $g \circ p_2$ that agree at either 1 or -1. Derive a contradiction.
3) Show that for any continuous map $F : \mathbb{S}^2 \longrightarrow \mathbb{R}^2$ there exists a point $x \in \mathbb{S}^2$ such that $F(x) = F(-x)$.
   Hint : if not, $x \mapsto \frac{F(x)-F(-x)}{\|F(x)-F(-x)\|}$ *maps* $\mathbb{S}^2$ *to* $\mathbb{S}^1$ *and restricts to an odd map from* $\mathbb{S}^1$ *to itself.*

This is the Borsuk-Ulam theorem in dimension 2.