Introduction to algebraic topology : Week 5

This week there are only three exercises ; you have to do them all for a maximum grade !

Exercise 1. Let G be the subgroup of the group of homeomorphisms of the plane to itself generated by the mappings $(x, y) \mapsto (x+1, y)$ and $(x, y) \mapsto (-x, y+1)$. Show that $\mathbb{R}^2 \longrightarrow \mathbb{R}^2/\mathrm{G}$ is a covering space. The quotient is the Klein Bottle.

If H is the subgroup of G generated by $(x, y) \mapsto (x+1, y)$ and $(x, y) \mapsto (x, y+2)$, show that $\mathbb{R}^2/\mathrm{H}$ identifies with a torus and defines a two-sheeted covering of the Klein bottle.

Exercise 2. Show that every covering morphism of connected covers over a locally connected base is itself a covering.

Exercise 3. Let $p : E \longrightarrow X$ be a covering, with E connected and X path-connected. Let $x \in X$ and set $F = p^{-1}(x)$. Explain briefly why $\mathrm{Aut}(E/X)$ can be viewed as a subgroup of $\mathfrak{S}_F$, the bijections of the set F.

Show that, under this identification, $\mathrm{Aut}(E/X)$ is exactly the subgroup of $\pi_1(X, x)$-equivariant bijections of F.