Introduction to algebraic topology : Week 7

Exercise 1. For each integer $n \geq 1$, compute the fundamental group of the real projective space $\mathbb{P}^n_{\mathbb{R}}$.

Exercise 2. If Γ is a group and $\alpha, \beta \in \Gamma$, define the commutator of α and β to be $[\alpha, \beta] = \alpha^{-1}\beta^{-1}\alpha\beta$. The commutator subgroup (also called the derived subgroup) of Γ is the subgroup generated by all the commutators of elements of Γ. It is denoted by $[\Gamma, \Gamma]$.

(a) Show that the commutator subgroup of any group is normal.
If Γ is a group we define $\Gamma_{ab} = \Gamma/[\Gamma, \Gamma]$

(b) What universal property does Γ_{ab} have ? Prove your answer.
Let X be a connected, locally path connected space that has a universal covering $\tilde{X}$. Fix $x \in X$. Define $\tilde{X}_{ab} = \tilde{X}/[\pi_1(X, x), \pi_1(X, x)] \longrightarrow X$.

(c) Show that $\tilde{X}_{ab} \longrightarrow X$ is a Galois covering of group $\pi_1(X, x)_{ab}$. The latter is called the abelian fundamental group.
A covering is said to be abelian if it is Galois with abelian Galois group.

(d) Show that any connected abelian covering of X has the form $\tilde{X}_{ab}/H \longrightarrow X$, for some subgroup H of $\pi_1(X, x)_{ab}$.

The same kind of things can be said about field extensions. You can try to work it out for yourself : define an abelian field extension, the abelian Galois group of a field and prove a result similar to (d).

Exercise 3. Show that every group can be realized as the Galois group of a galois covering.

Using corollary 1.28 in Hatcher, show that given a group G and a normal subgroup $N \subset G$, there exists a galois covering $p : E \longrightarrow X$ such that $\pi_1(X, x) = G$, $\pi_1(E, e) = N$ and $\mathrm{Aut}(E/X) = G/N$, for some $x \in X$ and $e \in p^{-1}(x)$.

Remark : these kind of problems are called inverse problems and are the motivation for what is called inverse Galois theory. One extremely important (and extremely wide open !) problem in number theory is the following : which groups can be realized as the Galois group of a finite extension of $\mathbb{Q}$? Using algebraic geometry and covering space theory, one can show that every finite group is the Galois group of a Galois extension of the field $\mathbb{C}(T)$.