Introduction to algebraic topology : Week 9

Exercise 1. 1) Let X be a path-connected (non-empty) topological space. Show that $\mathrm{H}_0(X) \simeq \mathbb{Z}$. Deduce $\mathrm{H}_0(X)$ when X is arbitrary.

2) Compute the singular homology groups of a discrete space.

Exercise 2. Let $\phi\colon \mathbf{C}_\bullet \longrightarrow \mathbf{D}_\bullet$ be a chain map of exact complexes. Suppose there exist two distinct residue classes modulo 3 such that ϕ_k is an isomorphism whenever k belongs to one of these two residue classes. Prove that ϕ_k is an isomorphism for all $k \in \mathbb{Z}$.

Exercise 3. Let $\phi, \phi'\colon \mathbf{C}_\bullet \to \mathbf{D}_\bullet$ be chain maps. A *chain homotopy* from ϕ to ϕ' is a collection homomorphisms $(P_k\colon \mathrm{C}_k \to \mathrm{D}_{k+1})_{k\in\mathbb{Z}}$ such that $\phi'_k - \phi_k = P_{k-1}\partial_k + \partial_{k+1}P_k$ for all $k \in \mathbb{Z}$.
(a) Prove that chain homotopy defines an equivalence relation on the set of chain maps from $\mathbf{C}_\bullet$ to $\mathbf{D}_\bullet$.
(b) Let $\phi, \phi'\colon \mathbf{C}_\bullet \to \mathbf{D}_\bullet$ and $\psi, \psi'\colon \mathbf{D}_\bullet \to \mathbf{E}_\bullet$ be chain homotopic. Prove that $\psi\phi, \psi'\phi'\colon \mathbf{C}_\bullet \to \mathbf{E}_\bullet$ are chain homotopic.
(c) Prove that chain homotopic maps induce the same maps on homology.