Leiden University

Introduction to Algebraic Topology

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Contents

Throughout these notes, maps are continuous maps and spaces are topological spaces. We denote the unit interval $[0,1]$ by I, and $\mathbb{N}$ is the set of natural numbers including zero.

1 The Fundamental Group

1.1 Homotopy and the Fundamental Group

Definition 1.1 (Homotopy)**.** Let X, Y be topological spaces. Two maps $f : X \to Y$ and $g : X \to Y$ are *homotopic* if there exists a map $H : X \times I \to Y$ such that $H(x, 0) = f(x)$ and $H(x, 1) = g(x)$ for all $x \in X$.

Remark 1.1. Homotopy defines an equivalence relation on $C^0(X, Y)$, which we will denote by $\sim$. We will use the $\simeq$ to denote homeomorphic spaces.

Fix a space X, and let $x_0 \in X$.

Definition 1.2 (Path)**.** A *path* in X is a map $\alpha : I \to X$.

Definition 1.3 (Loop)**.** A *loop* in X is a path α such that $\alpha(0) = \alpha(1)$.

Proposition 1.1. *The set $\pi_1(X, x_0)$ of loops at x_0 modulo the equivalence relation defined by homotopy, can be made into a group.*

Definition 1.4 (Path homotopy)**.** Two loops (at x_0) are *path-homotopic* if they are homotopic with base-preserving homotopy, i.e. $H(-, t)$ is a loop at x_0 for all $t \in I$.

Definition 1.5 (Concatenation)**.** Let α, β be loops in X. The *concatenation* $\alpha \odot \beta : I \to X$ of α and β is defined as

$$\begin{aligned} \alpha \odot \beta : I &\longrightarrow X \\ t &\longmapsto \begin{cases} \alpha(2t), & \text{if } t \in [0, \frac{1}{2}], \\ \beta(2t-1), & \text{if } t \in [\frac{1}{2}, 1]. \end{cases} \end{aligned} \tag{1}$$

Proposition 1.2. *Let $f : X \to Y$ be a map. There exists a group homomorphism*

$$\begin{aligned} f_* : \pi_1(X, x_0) &\longrightarrow \pi_1(Y, f(x_0)) \\ [\alpha] &\longmapsto [f \circ \alpha]. \end{aligned} \tag{2}$$

Proposition 1.3. *The map $\pi_1 : \mathbf{Top}_* \to \mathbf{Grp}$ from the category of pointed topological spaces to the category of groups is a functor.*

In short, a category $\mathcal{C}$ consists of "objects" and the appropriate maps between them. A functor is a map $F : \mathcal{C} \to \mathcal{D}$ from one category $\mathcal{C}$ to another category $\mathcal{D}$, which maps an object $c \in \mathcal{C}$ to an object $F(c) \in \mathcal{D}$, and a map $f : c \to c'$ between objects $c, c' \in \mathcal{C}$ to a map $F(f) : F(c) \to F(c')$ between objects $F(c), F(c') \in \mathcal{D}$.

Corollary 1.1. *Let X, Y be path connected spaces. Then $\pi_1(X) \ncong \pi_1(Y) \implies X \not\simeq Y$.*[1]

This corollary hints at the fact that functors "detect" non-homeomorphic spaces.

Examples 1.1.

- $\pi_1(\mathbb{S}^1, 1) = \mathbb{Z}$.
- $\pi_1(\mathbb{R}^m, x) = \{0\}$ for any $x \in \mathbb{R}^m$, for all $m \in \mathbb{N}$. It follows that $\mathbb{S}^1 \not\simeq \mathbb{R}^m$ for all $m \in \mathbb{N}$.
- $\pi_1(\mathbb{T}^2, x) = \mathbb{Z} \oplus \mathbb{Z}$ for any $x \in \mathbb{T}^2$, as a loop is determined by how many times it goes "around" the torus, and how many times "over" the torus.

Now given a space X and $x_0 \in X$, can we compute $\pi_1(X, x_0)$? If, for two spaces X and Y and some $y_0 \in Y$, we know that $\pi_1(X, x_0) = \pi_1(Y, y_0)$, we still cannot say much about X and Y.

Definition 1.6 (Homotopy equivalence)**.** Two spaces X, Y are *homotopy equivalent* if and only if there exist maps $f : X \to Y$ and $g : Y \to X$ such that $f \circ g \sim \mathrm{id}_Y$ and $g \circ f \sim \mathrm{id}_X$. The maps f and g are called *homotopy equivalences.*

Theorem 1.1. *If $f : X \to Y$ is a homotopy equivalence between spaces X and Y, and $x_0 \in X$, then $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(y_0)$ is an isomorphism.*

[1] For path connected spaces, the fundamental group does not depend on the base-point.

1.2 Deformation Retracts and the Fundamental Group

Let X be a space, let $A \subset X$ be a subspace, and let $\iota_A : A \to X$ be the inclusion map.

Definition 1.7 (Retraction)**.** A map $r : X \to A$ is a *retraction* if $r\big|_A = \mathrm{id}_A$. If there exists a retraction $r : X \to A$, then A is called a *retract* of X.

Definition 1.8 (Deformation retraction)**.** A map $r : X \to A$ is a *deformation retraction* if it is a retraction and $\iota_A \circ r \simeq \mathrm{id}_X$.

Remarks 1.1.

- A deformation retraction is a homotopy equivalence.

- If r is a retraction, then the identity on A factors as

$$\begin{array}{ccc} X & \xrightarrow{\quad r \quad} & A \\ & {\scriptstyle \iota_A} \nwarrow \quad \nearrow {\scriptstyle \mathrm{id}_A} & \\ & A & \end{array}$$

  so for $a \in A$ we have $r_* \circ (\iota_A)_* = \mathrm{id}_{\pi_1(A,a)}$. Then $(\iota_A)_*$ is injective and r_* is surjective, and $\mathrm{im}(r_*) \supset \mathrm{im}(r_* \circ (\iota_A)_*) = \pi_1(A, a)$.

Theorem 1.2 (Brouwer's fixed point theorem in $\mathbb{R}^2$)**.** *Let* $\overline{\mathbb{D}^2} := \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 1\}$ *be the closed unit disk in* $\mathbb{R}^2$*. Every map* $f : \overline{\mathbb{D}^2} \to \overline{\mathbb{D}^2}$ *has a fixed point.*

Proof. Suppose there exists a map $f : \overline{\mathbb{D}^2} \to \overline{\mathbb{D}^2}$ which has no fixed points. Since by assumption $f(x) \neq x$ holds for all $x \in \overline{\mathbb{D}^2}$, we can, for all $x \in \overline{\mathbb{D}^2}$, draw a line between x and $f(x)$, and consider the point of intersection of this line with $\mathbb{S}^1$ which is closest to x. This defines a retraction

$$r : \overline{\mathbb{D}^2} \to \mathbb{S}^1, \tag{3}$$

and the inclusion $\iota_{\mathbb{S}^1} : \mathbb{S}^1 \to \overline{\mathbb{D}^2}$ induces an injective homomorphism $(\iota_{\mathbb{S}^1})_* : \pi_1(\mathbb{S}^1, 1) \to \pi_1(\overline{\mathbb{D}^2}, 1)$. This is clearly a contradiction since[2] $\pi_1(\mathbb{S}^1, 1) = \mathbb{Z}$ and $\pi_1(\overline{\mathbb{D}^2}, 1) = \{0\}$. We conclude that every map $f : \overline{\mathbb{D}^2} \to \overline{\mathbb{D}^2}$ has a fixed point. □

Exercise 1.1. Show that for all $n \in \mathbb{N}$, every convex subset of $\mathbb{R}^n$ has trivial fundamental group.

Proof. Let $n \in \mathbb{N}$, let $C \subset \mathbb{R}^n$ be a convex subset, let $x_0 \in C$, and let $c_{x_0} : C \to \{x_0\}$ be the constant map with image $\{x_0\}$. Since C is convex, the map

$$\begin{aligned} H : C \times I &\longrightarrow C \\ (x, t) &\longmapsto t x_0 + (1 - t) x \end{aligned} \tag{4}$$

is a well-defined homotopy between id_X and c_{x_0}, so C is contractible and thus homotopy equivalent to a one-point space, so $\pi_1(C, x_0) = \{0\}$. The conclusion follows. □

Theorem 1.3. *Two spaces* X, Y *are homotopically equivalent if and only if there exists a space* Z *such that both* X *and* Y *are deformation retracts of* Z*.*

Proof. If X and Y are both deformation retracts of some space Z, then $X \simeq Z$ and $Y \simeq Z$, so by transitivity we have $X \simeq Y$.

Suppose that X and Y are homotopically equivalent. We make the following definition.

Definition 1.9 (Mapping cylinder)**.** Let X, Y be spaces, and let $f : X \to Y$ be a map. The *mapping cylinder* is defined as

$$Z_f := \frac{Y \sqcup (X \times I)}{\forall x \in X : \; f(x) \sim (x, 0)} \tag{5}$$

Remark 1.2. Let $q : Y \sqcup (X \times I) \to Z_f$ be the quotient map. It is clear that $\tilde{Y} := q[Y] \simeq Y$ and $\tilde{X} := q[X \times \{1\}] \simeq X$.

[2] Note that the disk is contractible, so $\pi_1(\overline{\mathbb{D}^2}, x) = \{0\}$ for all $x \in \overline{\mathbb{D}^2}$.

Define the map[3]

$$\begin{aligned} A : Z_f &\longrightarrow Z_f \\ \overline{(x,s)} &\longmapsto \overline{(x,0)} \\ \overline{y} &\longmapsto \overline{y}. \end{aligned} \tag{6}$$

Then[4]

$$\begin{aligned} H_1 : Z_f \times I &\longrightarrow Z_f \\ (\overline{(x,s)}, t) &\longmapsto \overline{(x, s(1-t))} \\ (\overline{y}, t) &\longmapsto \overline{y} \end{aligned} \tag{7}$$

is clearly a homotopy between id_{Z_f} and A, so $A \simeq \mathrm{id}_{Z_f}$. Clearly $A\big|_{\tilde{Y}} = \mathrm{id}_{\tilde{Y}}$, so $\tilde{Y}$ is a deformation retract of Z_f. Suppose $f : X \to Y$ and $g : Y \to X$ are maps such that $f \circ g \simeq \mathrm{id}_Y$ and $g \circ f \simeq \mathrm{id}_X$, with homotopies F and G, respectively. Define the maps

$$\begin{aligned} B : Z_f &\longrightarrow Z_f \\ \overline{(x,s)} &\longmapsto \overline{(g(f(x)),0)} \\ \overline{y} &\longmapsto \overline{(g(y),0)} \end{aligned} \tag{8}$$

and

$$\begin{aligned} H_2 : Z_f \times I &\longrightarrow Z_f \\ (\overline{(x,s)}, t) &\longmapsto \overline{F(f(x), 1-t)}, \\ (\overline{y}, t) &\longmapsto \overline{F(y, 1-t)}. \end{aligned} \tag{9}$$

For all $\overline{(x,s)}, \overline{y} \in Z_f$ we have

$$\begin{aligned} H_2((\overline{(x,s)}, 0)) &= \overline{F(f(x),1)} \\ &= \overline{f(x)} \\ &= \overline{(x,0)}, \end{aligned} \qquad \begin{aligned} H_2((\overline{y}, 0)) &= \overline{F(y,1)} \\ &= \overline{\mathrm{id}_Y(y)} \\ &= \overline{y}, \end{aligned} \tag{10}$$

and

$$\begin{aligned} H_2((\overline{(x,s)}, 1)) &= \overline{F(f(x),0)} \\ &= \overline{f(g(f(x)))} \\ &= \overline{(g(f(x)),0)}, \end{aligned} \qquad \begin{aligned} H_2((\overline{y}, 1)) &= \overline{F(y,0)} \\ &= \overline{f(g(y))} \\ &= \overline{(g(y),0)}, \end{aligned} \tag{11}$$

so $H_2(z,0) = A(z)$ and $H_2(z,1) = B(z)$ for all $z \in Z_f$. Lastly, define the maps

$$\begin{aligned} C : Z_f &\longrightarrow Z_f \\ \overline{(x,s)} &\longmapsto \overline{(G(x,s),1)} \\ \overline{y} &\longmapsto \overline{(g(y),1)} \end{aligned} \tag{12}$$

and

$$\begin{aligned} H_3 : Z_f \times I &\longrightarrow Z_f \\ (\overline{(x,s)}, t) &\longmapsto \overline{(G(x,st),t)} \\ (\overline{y}, t) &\longmapsto \overline{(g(y),t)}. \end{aligned} \tag{13}$$

For all $\overline{(x,s)}, \overline{y} \in Z_f$ we have

$$\begin{aligned} H_3((\overline{(x,s)}, 0)) &= \overline{(G(x,0),0)} \\ &= \overline{(g(f(x)),0)}, \end{aligned} \qquad H_3((\overline{y}, 0)) = \overline{(g(y),0)}, \tag{14}$$

[3]Note that A is well-defined and continuous, since the map $\tilde{A} : Y \sqcup (X \times I) \to Z_f$ defined by $\tilde{A}((x,s)) = \overline{(x,0)}$ and $\tilde{A}(y) = \overline{y}$ for all $(x,s) \in X \times I$ and for all $y \in Y$ is continuous and respects the equivalence relation used to define Z_f, i.e. for all $x \in X$ we have $\tilde{A}((x,0)) = \overline{(x,0)} = \overline{f(x)} = \tilde{A}(f(x))$ (it suffices to check only these elements, since these are the only elements identified in Z_f). By the universal property of the quotient topology, $\tilde{A}$ descends to the quotient, implying that A is continuous.

[4]The proofs that this and all other maps in this proof are well-defined and continuous are straightforward verifications similar to footnote 1.4, and are omitted.

and

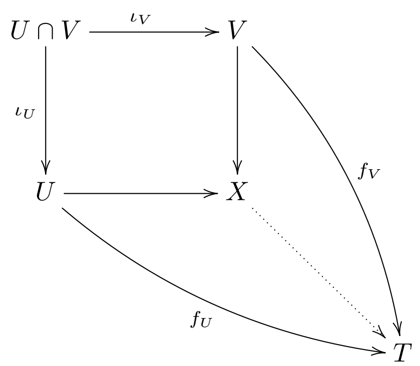
$$H_3((\overline{(x,s)},1)) = \overline{(G(x,s),1)}, \qquad H_3((\overline{y},1)) = \overline{(g(y),1)}, \tag{15}$$

so $H_3(z,0) = B(z)$ and $H_3(z,1) = C(z)$ for all $z \in Z_f$. It follows that $\mathrm{id}_{Z_f} \simeq A \simeq B \simeq C$, so $C \simeq \mathrm{id}_{Z_f}$. We clearly have $C[Z_f] \subset \tilde{X}$, and since $G(x,1) = x$ holds for all $x \in X$, we have $C(\overline{(x,1)}) = \overline{(x,1)}$ for all $x \in X$ and thus $C\big|_{\tilde{X}} = \mathrm{id}_{\tilde{X}}$, which shows that $\tilde{X}$ is a deformation retract of Z_f. □

1.3 The Seifert-Van Kampen Theorem

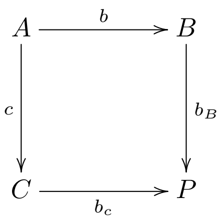
Let X be a space, let U, V be open subsets of X such that $X = U \cup V$, and let $w \in U \cap V$.

Question 1.1. *In terms of $\pi_1(U,w)$, $\pi_1(V,w)$ and $\pi_1(U \cap V, w)$, what should $\pi_1(X,w)$ be? If we have a commutative diagram*

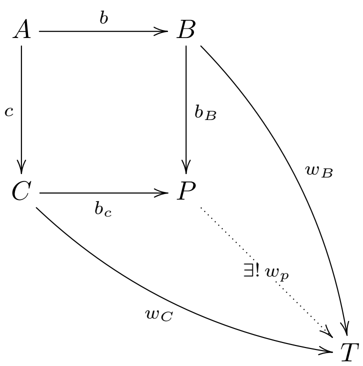


then $f_V(x) = f_U(x)$ for all $x \in U \cap V$, so $f_V\big|_{U\cap V} = f_U\big|_{U \cap V}$, and we can glue the maps f_V and f_W to a map from X to T.

Definition 1.10 (Pushout)**.** Let $\mathcal{C}$ be a category, let $A, B, C, P \in \mathcal{C}$ be objects, with maps $b : A \to B$ and $c : A \to C$. A diagram



is called a *pushout* of (b, c) if it commutes and for all objects $T \in \mathcal{C}$ and maps $w_B : B \to T$ and $w_C : C \to T$ such that the diagram



commutes, there exists a unique map $w_p : P \to T$ (such that the resulting diagram commutes as well).

Remarks 1.2.

- Not all categories have pushouts.
- A pushout is unique up to unique isomorphism.

Question 1.2. *Do the categories* **Set**, **Grp**, **Ring** *and* **R-mod** *have pushouts, and if they do, what are they?*

Question 1.3. *Let $\mathcal{C}$ and $\mathcal{D}$ be categories, let $A, B, C \in \mathcal{C}$ be objects with maps $b : A \to B$ and $c : A \to C$, and let $F : \mathcal{C} \to \mathcal{D}$ be a functor, then F sends*

$$\begin{array}{ccc} A & \xrightarrow{b} & B \\ {\scriptstyle c}\downarrow & & \\ C & & \end{array} \qquad \text{to} \qquad \begin{array}{ccc} F(A) & \xrightarrow{F(b)} & F(B) \\ {\scriptstyle F(c)}\downarrow & & \\ F(C) & & \end{array}$$

Does F preserve pushouts?

Theorem 1.4 (Seifert-Van Kampen)**.** *Let X be a connected space. Let U, V be path connected open subsets of X such that $X = U \cup V$ and $U \cap V$ is path connected. Let $w \in U \cap V$, and let $i : U \cap V \to U$, $j : U \cap V \to V$, $k : U \to X$ and $l : V \to X$ be the inclusions. Then*

$$\begin{array}{ccc} \pi_1(U \cap V, w) & \xrightarrow{i_*} & \pi_1(U, w) \\ {\scriptstyle j_*}\downarrow & & \downarrow{\scriptstyle k_*} \\ \pi_1(V, w) & \xrightarrow[l_*]{} & \pi_1(X, w) \end{array}$$

is a pushout in **Grp**.

Question 1.4. *What are the pushouts in* **Grp***?*

Let H, G be groups, and let K be a subgroup of both H and G. Suppose $K = \{1\}$, let T be a group and let $f_G : G \to T$ and $f_H : H \to T$ be group homomorphisms such that $f_G \circ \iota_G = f_H \circ \iota_H$, where ι_G and ι_H are the inclusions. In this case, there are no restrictions on f_G and f_H, since agreeing on the trivial subgroup is immediate as f_G and f_H are group homomorphisms.

Definition 1.11 (Free product)**.** Let H and G be groups. The *free product* of H and G is the free group $G \star H$ consisting of all reduced words formed with elements from G and H, i.e. $\{g_1 h_1 \dots g_n h_n : n \in \mathbb{N}, g_i \in G, h_i \in H\}$.

Example 1.1.

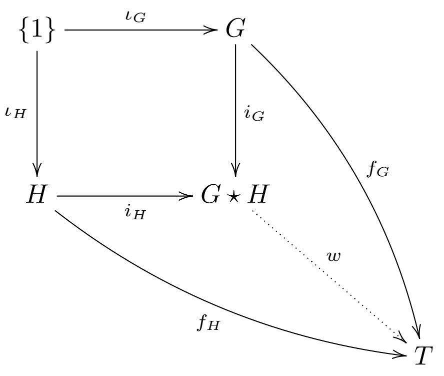
- $\mathbb{Z} \star \mathbb{Z} = \langle x \rangle \star \langle y \rangle = \{x^{a_1} y^{b_1} \dots x^{a_n} y^{b_n} : n \in \mathbb{N}, a_i, b_i \in \mathbb{N}\}$.
- For $n \in \mathbb{N}$, $F_n := \underbrace{\mathbb{Z} \star \dots \star \mathbb{Z}}_{n \text{ times}}$ is the free group on n generators.

Remark 1.3. Every group with two generators is a quotient of $\mathbb{Z} \star \mathbb{Z}$.

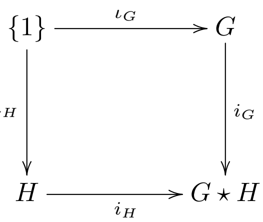
Let H and G be groups, and let $\iota_G : \{1\} \to G$, $\iota_H : \{1\} \to H$, $i_G : G \to G \star H$ and $i_H : H \to G \star H$ be the inclusions. Let T be another group, and let $f_G : G \to T$ and $f_H : H \to T$ be any group homomorphisms. Then

$$\begin{aligned} w : G \star H &\longrightarrow T \\ g_1 h_1 \dots g_n h_n &\longmapsto f_G(g_1) f_H(h_1) \dots f_G(g_n) f_H(h_n) \end{aligned} \tag{16}$$

is a well-defined group homomorphism such that



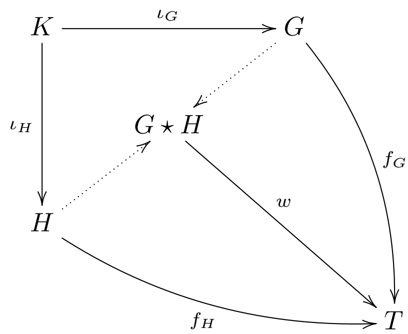
commutes, which by the universal property of the free product implies that



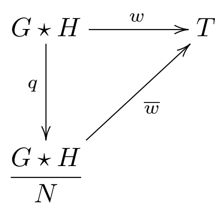
is a pushout in **Grp**.

Corollary 1.2. *In the situation of* **Theorem 1.4**, *assume that $U \cap V$ is simply connected. Then* $\pi_1(X, w) = \pi_1(U, w) \star \pi_1(V, w)$.

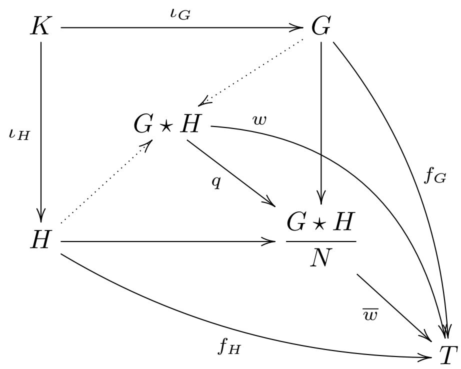
Now let K be a subgroup of both H and G, and suppose that the diagram



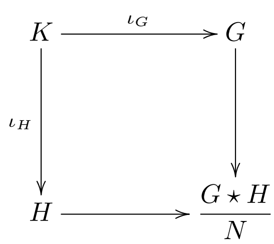
commutes, where now ι_G and ι_H stand for the inclusions of K into G and H, respectively. Since $w(\iota_G(k)) = w(\iota_H(k))$ for all $k \in K$, we have $w(\iota_G(k)\iota_H(k)^{-1}) = 1$ for all $k \in K$. Let N be the normal closure of the set $\langle \iota_G(k)\iota_H(k)^{-1} : k \in K \rangle$, then $N \subset \ker(w)$ and thus there exists a unique homomorphism $\overline{w}$ making the diagram



commute[5], so the diagram



commutes, which implies that



is a pushout in **Grp**.

[5] Here q is of course the quotient map.

Definition 1.12 (Amalgamated free product)**.** Following the construction given above, the group

$$G \underset{K}{\star} H := \frac{G \star H}{N} \tag{17}$$

is the *amalgamated free product* of G and H via K.

We can now state **Theorem 1.4** more explicitly.

Theorem 1.5 (Seifert-Van Kampen)**.** *In the situation of* **Theorem 1.4** *we have*

$$\pi_1(X, w) = \pi_1(U, w) \underset{\pi_1(U \cap V, w)}{\star} \pi_1(V, w). \tag{18}$$

Example 1.2. Let $n \geq 2$. Let $\mathcal{N}$ and $\mathcal{S}$ be the north- and southpole of $\mathbb{S}^n$, respectively, and define the open subsets $U := \mathbb{S}^n \setminus \{\mathcal{N}\}$ and $V := \mathbb{S}^n \setminus \{\mathcal{S}\}$ of $\mathbb{S}^n$. Then U, V and $U \cap V$ are path connected, and since U and V are homeomorphic to $\mathbb{R}^n$, U and V are simply connected, so by **Theorem 1.5** we have $\pi_1(\mathbb{S}^n, x) = \{1\}$ for all $x \in \mathbb{S}^n$.

1.3.1 Wedge sum

Another example of a pushout is the *wedge sum.* Let (X_1, p_1) and (X_2, p_2) be pointed manifolds.

Definition 1.13 (Wedge sum)**.** The *wedge sum* of (X_1, p_1) and (X_2, p_2) is the quotient space

$$X_1 \vee X_2 := \frac{X_1 \sqcup X_2}{p_1 \sim p_2}. \tag{19}$$

Lemma 1.1. *Let $q \in X_1 \vee X_2$ be the point $\overline{p_1} = \overline{p_2}$. Then*

$$\begin{array}{ccc} \{*\} & \longrightarrow & (X_1, p_1) \\ \downarrow & & \downarrow \\ (X_2, p_2) & \longrightarrow & (X_1 \vee X_2, q) \end{array}$$

is a pushout in $\mathbf{Man}_*$*, the category of pointed topological manifolds.*

Proposition 1.4. *In the situation of* **Lemma 1.1***, we have*

$$\pi_1(X_1 \vee X_2, q) = \pi_1(X_1, p_1) \star \pi_1(X_2, p_2). \tag{20}$$

Proof. Let[6] W_1 and W_2 be neighbourhoods of p_1 and p_2 respectively such that W_i deformation retracts onto p_i for $i \in \{1, 2\}$. Let $q : X_1 \sqcup X_2 \to X_1 \vee X_2$ be the quotient map, and consider $U := q[X_1 \sqcup W_2]$ and $V := q[X_2 \sqcup W_1]$. Then U and V are open in $X_1 \vee X_2$ and $U \cup V = X_1 \vee X_2$, so by **Theorem 1.5** we have

$$\pi_1(X_1 \vee X_2, q) = \pi_1(U, p_1) \underset{\pi_1(U \cap V, q)}{\star} \pi_1(V, p_2). \tag{21}$$

Since the inclusions $\{*\} \to U \cap V$, $q[X_1] \to U$ and $q[X_2] \to V$ are all homotopy equivalences, it follows that

$$\pi_1(X_1 \vee X_2, q) = \pi_1(X_1, p_1) \star \pi_1(X_2, p_2). \tag{22}$$

□

[6] This works as long as we can find contractible subspaces.

2 Covering Space Theory

2.1 Covering Spaces

2.1.1 Fibre bundles

Definition 2.1 (Fibre bundle)**.** A *fibre bundle* is a quadruple $\xi = (E, B, p, F)$ such that:

- E, B, and F are spaces, and
- $p : E \to B$ is a map, and
- for all $b \in B$ there exists a neighbourhood $U_b \subset B$ of b and a homeomorphism $\Phi_b : p^{-1}[U_b] \to U_b \times F$ such that

$$\begin{array}{ccc} p^{-1}[U_b] & \xrightarrow{\Phi} & U_b \times F \\ {\scriptstyle p}\downarrow & \swarrow{\scriptstyle \mathrm{proj}_1} & \\ U_b & & \end{array}$$

  commutes.

E is called the *total space* of ξ, B is called the *base* of ξ, and F is called the *fibre* of ξ.

Remark 2.1. A fibre bundle is said to be trivial if $E \simeq B \times F$.

Definition 2.2 (Map between fibre bundles)**.** Let $\xi_1 = (E_1, B_1, p_1, F_1)$ and $\xi_1 = (E_2, B_2, p_2, F_2)$ be fibre bundles. A *map of fibre bundles* $\theta : \xi_1 \to \xi_2$ is a pair (φ, ψ) such that

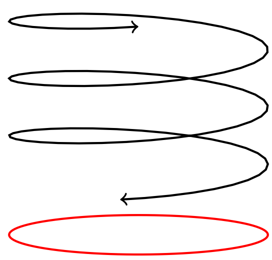
$$\begin{array}{ccc} E_1 & \xrightarrow{\varphi} & E_2 \\ {\scriptstyle p_1}\downarrow & & \downarrow{\scriptstyle p_2} \\ B_1 & \xrightarrow[\psi]{} & B_2 \end{array}$$

commutes.

Remark 2.2. If $p^{-1}(b)$ is a vector space for all $b \in B$, then a fibre bundle is called a vector bundle.

Definition 2.3 (Covering space)**.** A *covering space* is a fibre bundle with discrete fibre, i.e. a map $p : E \to B$ such that each $b \in B$ has a neighbourhood U_b with $p^{-1}[U_b] = \sqcup_{\alpha \in L} V_\alpha$ for some indexing set L, where V_α is homeomorphic to U_b for each $\alpha \in L$. This means that p is a local homeomorphsim; p is called a *covering map.*

Example 2.1 (Mother of examples)**.** The map $\varepsilon : \mathbb{R} \to \mathbb{S}^1$ sending $t \in \mathbb{R}$ to $e^{2\pi i t}$ is a covering map.



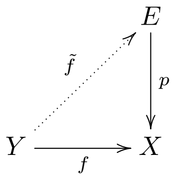
Definition 2.4 (Evenly covered subset)**.** Let $p : E \to B$ be a covering map. An open subset $U \subset B$ is said to be *evenly covered* if $p^{-1}[U]$ is homeomorphic to a disjoint union $\sqcup_{\alpha \in L} V_\alpha$ of open subsets $V_\alpha \subset E$, for some indexing set L, such that V_α is homeomorphic to U for each $\alpha \in L$.

Definition 2.5 (Local section)**.** Let $p : E \to B$ be a covering. A *local section* for p is map $s : U \to E$, where U is a non-empty open subset of B, and such that $p \circ s = \mathrm{id}_U$.

Proposition 2.1. *Covering spaces have local sections.*

2.1.2 Lifting problems

Question 2.1. *Let $p : E \to X$ be a covering, let Y be a space and let $f : Y \to X$ be a map. Does there exist a map $\tilde{f} : Y \to E$ (this map is called a* lift *of f) such that the diagram*



commutes? If it exists, is it unique?

Lemma 2.1 (Unique lifting property)**.** *In this situation, suppose $\tilde{f}_1$ and $\tilde{f}_2$ are two lifts of f. If Y is connected and there exists some $y \in Y$ such that $\tilde{f}_1(y) = \tilde{f}_2(y)$, then $\tilde{f}_1 = \tilde{f}_2$.*

Proof. Define $S := \{y \in Y : \tilde{f}_1(y) = \tilde{f}_2(y)\} \subset Y$. Note that by assumption, S is non-empty. Let $s \in S$, and let $N \subset X$ be an evenly covered neighbourhood of $f(s)$. Let $\alpha_0 \in L$ be such that N_{α_0} contains $\tilde{f}_1(s) = \tilde{f}_2(s)$, and define the open subset $V := \tilde{f}_1^{-1}[N_{\alpha_0}] \cap \tilde{f}_2^{-1}[N_{\alpha_0}] \subset Y$. Then V is a neighbourhood of s such that $\tilde{f}_1[V] \subset N_{\alpha_0}$ and $\tilde{f}_2[V] \subset N_{\alpha_0}$, and since p restricted to N_{α_0} is homeomorphism onto N and $p \circ \tilde{f}_1 = p \circ \tilde{f}_2$, $\tilde{f}_1$ and $\tilde{f}_2$ agree on V, so $V \subset S$, which implies that S is open.

Let $y \in Y \setminus S$, and let $N \subset X$ be a neighbourhood of $(p \circ \tilde{f}_1)(y) = (p \circ \tilde{f}_2)(y)$. Let $\alpha_1, \alpha_2 \in L$ such that $\tilde{f}_1(y) \in N_{\alpha_1}$ and $\tilde{f}_2(y) \in N_{\alpha_2}$, then[7] $N_{\alpha_1} \cap N_{\alpha_2} = \emptyset$. Define $V := \tilde{f}_1^{-1}[N_{\alpha_1}] \cap \tilde{f}_2^{-1}[N_{\alpha_2}] \subset Y$, then $\tilde{f}_1[V] \subset N_{\alpha_1}$ and $\tilde{f}_2[V] \subset N_{\alpha_2}$, so $\tilde{f}_1$ and $\tilde{f}_2$ don't agree on V, so $V \subset Y \setminus S$. This shows that $Y \subset S$ is open, so S is a non-empty open and closed subset of Y, which by the connectedness of Y implies that $Y = S$. This shows that $\tilde{f}_1$ equals $\tilde{f}_2$. □

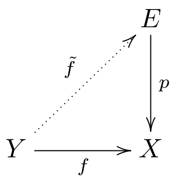
Recall the following two theorems from *Runde* [1].

Theorem 2.1 (Path lifting)**.** *Let $p : E \to X$ be a covering, and let $\gamma : [0,1] \to X$ be a path in X. Let $e_0 \in p^{-1}(\gamma(0))$. There exists a unique lift $\tilde{\gamma}_{e_0} : [0,1] \to E$ such that $\tilde{\gamma}_{e_0}(0) = e_0$ and $p \circ \tilde{\gamma}_{e_0} = \gamma$.*

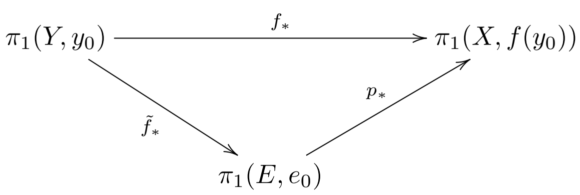
Theorem 2.2 (Homotopic path lifting)**.** *Let $p : E \to X$ be a covering. Let α, β be paths in X such that $\alpha(0) = \beta(0)$ and $\alpha(1) = \beta(1)$, and let $e_0 \in p^{-1}(\gamma(0))$. Then $\tilde{\alpha}_{e_0} \sim \tilde{\beta}_{e_0}$ if and only if $\alpha \sim \beta$.*

Exercise 2.1. Let $p : E \to X$ be a (connected) covering, and let $e \in E$. Show that the induced group homomorphism $p_* : \pi_1(E, e) \to \pi_1(X, p(e))$ is injective.

Theorem 2.3 (Lifting criterion)**.** *Let $p : E \to X$ be a covering, let Y be a connected and locally path connected space, and let $f : Y \to X$ be a map. Let $y_0 \in Y$ and $e_0 \in E$ such that $p(e_0) = f(y_0)$. Then f has a lift $\tilde{f} : Y \to E$ such that $\tilde{f}(y_0) = e_0$ if and only if $f_*[\pi_1(Y, y_0)] \subset p_*[\pi_1(E, e_0)]$.*



Proof. Suppose there exists such a a lift $\tilde{f}$. Then $p \circ \tilde{f} = f$, so the diagram



commutes, i.e. $f_*[\pi_1(Y, y_0)] \subset p_*[\pi_1(E, e_0)]$.

Let $y \in Y$, and let[8] α be a path from y_0 to y. Define $\tilde{f} : Y \to E$ by $\tilde{f}(y) := (\widetilde{f \circ \alpha})_{e_0}(1)$ for all $y \in Y$, where $(\widetilde{f \circ \alpha})_{e_0}$ is the lift of $f \circ \alpha$ at e_0 (which exists by **Theorem 2.1**). By definition, we have $p \circ \tilde{f} = f$ and $\tilde{f}(y_0) = e_0$. We need to prove that $\tilde{f}$ is well-defined and continuous.

[7] Since p restricted to N_{α_i} is a homeomorphism onto N for $i \in \{1, 2\}$

[8] Since Y is connected and locally path connected, it is also path connected.

Let α, β be paths from y_0 to y. Define

$$\begin{aligned} \beta \odot \overline{\alpha} : [0,1] &\longrightarrow Y \\ t &\longmapsto \begin{cases} \beta(2t), & \text{if } t \in [0, \frac{1}{2}], \\ \alpha(2-2t), & \text{if } t \in [\frac{1}{2}, 1], \end{cases} \end{aligned} \tag{23}$$

then $\beta \odot \overline{\alpha}$ is a loop at y_0, so $f \circ (\beta \odot \overline{\alpha})$ is a loop at $f(y_0) = p(e_0)$. Since by assumption the inclusion $f_*[\pi_1(Y, y_0)] \subset p_*[\pi_1(E, e_0)]$ holds, there exists a loop τ in E at e_0 such that $[f \circ \beta \odot \overline{\alpha}] = [p \circ \tau]$, so there exists a homotopy H such that $f \circ (\beta \odot \overline{\alpha}) \sim_H p \circ \tau$, i.e. such that $(f \circ \beta) \odot (\overline{f \circ \alpha}) \sim_H p \circ \tau$. Then $(f \circ \beta) \sim_H (p \circ \tau) \odot (f \circ \alpha)$, and since the lifts

$$(\widetilde{f \circ \beta})_{e_0}, ((p \circ \widetilde{\tau) \odot (f} \circ \alpha))_{e_0} \tag{24}$$

exist by **Theorem 2.1** and are homotopic by **Theorem 2.2**, their endpoints are equal. Note that

$$((p \circ \widetilde{\tau) \odot (f} \circ \alpha))_{e_0} = \tau \odot (\widetilde{f \circ \alpha})_{e_0}, \tag{25}$$

and since τ is a loop at e_0 we have

$$(\widetilde{f \circ \alpha})_{e_0}(1) = (\tau \odot (\widetilde{f \circ \alpha}))_{e_0}(1), \tag{26}$$

which means that

$$(\widetilde{f \circ \alpha})_{e_0}(1) = (\widetilde{f \circ \beta})_{e_0}(1), \tag{27}$$

so $\tilde{f}$ is well-defined.

Let $y \in Y$, let $U \subset X$ be an evenly covered neighbourhood of $f(y)$, and let $\tilde{U} \subset E$ be the sheet containing $\tilde{f}(y)$. Note that $\tilde{U}$ is a connected component of $p^{-1}[U]$. Let V be the path component of $f^{-1}[U]$ containing y. As Y is locally path connected, V is open. Let $\tilde{y}_1, \tilde{y}_2 \in \tilde{f}[V]$ and let $y_1, y_2 \in Y$ such that $\tilde{f}(y_1) = \tilde{y}_1$ and $\tilde{f}(y_2) = \tilde{y}_2$. Let α be a path from y_0 to y_1 in Y and let β be a path from y_1 to y_2 in V, then $f \circ (\alpha \odot \beta)$ is a path in E from e_0 to $\tilde{f}(y_2)$ via $\tilde{f}(y_1)$; in particular, $f \circ \beta$ is a path in $\tilde{f}[Y]$ from $\tilde{f}(y_1)$ to $\tilde{f}(y_2)$, so $\tilde{f}[V]$ is path connected and thus connected. Since $\tilde{f}[V] \cap \tilde{U} \neq \emptyset$ (as $\tilde{f}(y) \in \tilde{f}[V] \cap \tilde{U}$) and $\tilde{f}[V]$ is connected we have $\tilde{f}[V] \subset \tilde{U}$. Then

$$p\big|_{\tilde{U}} \circ \tilde{f}\big|_V = f\big|_V, \tag{28}$$

and since $p\big|_{\tilde{U}}$ is a homeomorphism, we have

$$\tilde{f}\big|_V = \left(p\big|_{\tilde{U}}\right)^{-1} \circ f\big|_V, \tag{29}$$

so $\tilde{f}$ restricted to V is continuous as a composition of continuous maps, and as y was arbitrary, this implies that $\tilde{f}$ is continuous. □

2.2 The Monodromy Action

We assume that all spaces are locally path connected.

Proposition 2.2. *Let $p : E \to X$ be a covering, and let $x \in X$. The map*

$$\begin{aligned} \mathcal{M} : \pi_1(X, x) \times p^{-1}(x) &\longrightarrow p^{-1}(x) \\ ([\alpha], e) &\longmapsto [\alpha] \cdot e = \tilde{\alpha}_e(1) \end{aligned} \tag{30}$$

is a group action of $\pi_1(X, x)$ on $p^{-1}(x)$.

Proof. By **Theorem 2.2**, this action is well-defined. For all $e \in p^{-1}(x)$ we have

$$\begin{aligned} [c_x] \cdot e &= c_e(1) \\ &= e, \end{aligned} \tag{31}$$

so the constant path at x acts as the identity element. For all $[\alpha], [\beta] \in \pi_1(X, x)$ we have

$$\begin{aligned} ([\alpha] \cdot [\beta]) \cdot e &= [\alpha \odot \beta] \cdot e \\ &= (\widetilde{\alpha \odot \beta})_e(1) \\ &= \tilde{\beta}_{\tilde{\alpha}_e(1)}(1) \\ &= [\alpha] \cdot ([\beta] \cdot e) \end{aligned} \tag{32}$$

for all $e \in p^{-1}(x)$, so the action is associative. □

Remarks 2.1. Let G be a group acting on a set S.

- The action is transitive if for all $s, s' \in S$ there exists a $g \in G$ such that $s' = g \cdot s$.
- The action is free if for all $s \in S$ we have $\text{Stab}_s = \{g \in G : g \cdot s = s\} = \{1\}$. Let $s_0 \in S$.
- If the action is regular (i.e. free and transitive), then for all $s \in S$ there exists a unique element $g = g(s_0, s) \in G$ such that $s = g(s_0, s) \cdot s_0$, so the map $P : S \to G$ sending s to $g(s_0, s)$ is a bijection (of sets).
- Let $s, s' \in S$. If G acts transitively on S, then Stab_s is conjugated to $\text{Stab}_{s'}$. Indeed, since G acts transitively on S, there exists an element $g \in G$ such that $s' = g \cdot s$. Let $h \in \text{Stab}_{s'}$, then $g \cdot s = s' = h \cdot s' = hg \cdot s$, so $s = g^{-1}hg \cdot s \implies g^{-1}hg \in \text{Stab}_s$, so $g^{-1}\text{Stab}_{s'}g \subset \text{Stab}_s$. Let $h \in \text{Stab}_s$, then since $s = g^{-1} \cdot s'$ we have $g^{-1} \cdot s' = s = h \cdot s = hg^{-1} \cdot s'$, so we have $s' = ghg^{-1} \cdot s'$, i.e. $ghg^{-1} = h'$ for some $h' \in \text{Stab}_{s'}$, so $h = g^{-1}h'g$ and thus $h \in g^{-1}\text{Stab}_{s'}g$, so $g^{-1}\text{Stab}_{s'}g = \text{Stab}_s$.
- Let $s \in S$ and $g \in G$, then $g^{-1}\text{Stab}_s g = \text{Stab}_{g^{-1}s}$. Let $h \in \text{Stab}_s$, let $g \in G$, and define $f := g^{-1}hg \in g^{-1}\text{Stab}_s g$ and $s' := g^{-1}s$. Then $fs' = g^{-1}hgg^{-1}s = g^{-1}hs = g^{-1}s = s'$, so $g^{-1}hg \in \text{Stab}_{g^{-1}\cdot s}$, so $g^{-1}\text{Stab}_s g \subset \text{Stab}_{g^{-1}\cdot s}$. Let $h \in \text{Stab}_{g^{-1}\cdot s}$, then $hg^{-1} \cdot s = g^{-1} \cdot s$, so $ghg^{-1} \cdot s = s$, so $ghg^{-1} = h' \in \text{Stab}_s$ and thus $h \in g^{-1}\text{Stab}_s g$, so $g^{-1}\text{Stab}_s g = \text{Stab}_{g^{-1}\cdot s}$.
- If G also acts on another set $\tilde{S}$, then $\varphi : S \to \tilde{S}$ is an G-equivariant map if and only if for all $s \in S$ and for all $g \in G$ we have $\varphi(g \cdot s) = g \cdot \varphi(s)$. We say that φ "respects" the action of G, and we get a category G-**Sets**.

Proposition 2.3. *If $p : E \to X$ is a connected covering, then the monodromy action is transitive.*

Proof. Let $e, e' \in p^{-1}(x)$. Since E is path connected[9], there exists a path $\gamma : [0, 1] \to E$ such that $\gamma(0) = e$ and $\gamma(1) = e'$. Then $[p \circ \gamma] \cdot e = \gamma(1) = e'$, so $\mathcal{M}$ is transitive. □

From now on, coverings are assumed to be connected, so orbits are the whole fibre.

Proposition 2.4. *Let $p : E \to X$ be a covering, let $x \in X$, and let $e \in p^{-1}(x)$ be in the fibre of x. Then* $\text{Stab}_e = p_*[\pi_1(E, e)] \subset \pi_1(X, x)$.

Proof. Let $[\alpha] \in \text{Stab}_e$. Then $[\alpha] \cdot e = e \iff \tilde{\alpha}_e(1) = e$, so $\tilde{\alpha}_e$ is a loop in E at e, hence $[\tilde{\alpha}_e] \in \pi_1(E, e)$. It follows that $[p \circ \tilde{\alpha}_e] = [\alpha] = p_*[\tilde{\alpha}_e] \in p_*[\pi_1(E, e)]$, so $\text{Stab}_e \subset p_*[\pi_1(E, e)]$.

Let $[\beta] \in \pi_1(E, e)$, then $[p \circ \beta] \cdot e = \beta(1) = e$, so $p_*[\pi_1(E, e)] \subset \text{Stab}_e$. Since E is connected, p_* is injective, and it follows that $\text{Stab}_e = p_*[\pi_1(E, e)] \subset \pi_1(X, x)$. □

Corollary 2.1.

1. *The monodromy action $\mathcal{M}$ is free on all fibres if and only E is simply connected.*
2. *If E is simply connected, then the sets $p^{-1}(x)$ and $\pi_1(X, x)$ are in bijection for all $x \in X$.*

Proposition 2.5. *Let $p : E \to X$ be a (connected) covering. Then for all $x \in X$, $p_*[\pi_1(E, e)]$ and $p_*[\pi_1(E, e')]$ are conjugated for all $e, e' \in p^{-1}(x)$. The groups $p_*[\pi_1(E, e)]$, as e varies over $p^{-1}(x)$, describe exactly one conjugacy class in $\pi_1(X, x)$.*

Definition 2.6 (Normal subgroup)**.** Let G be a a group, and let $H \subset G$ be a subgroup. If $g^{-1}Hg = H$ for all $g \in G$, then H is a *normal subgroup* of G.

Proposition 2.6. *Let $p : E \to X$ be a (connected) covering. Then $p_*[\pi_1(E, e)]$ is a normal subgroup of $\pi_1(X, p(e))$ for some $e \in E$ if and only if $p_*[\pi_1(E, e)]$ is a normal subgroup of $\pi_1(X, p(e))$ for all $e \in E$.*

Definition 2.7 (Normal covering)**.** A (connected) covering $p : E \to X$ for which **Proposition 2.6** holds is called a *normal* (*regular*, *Galois*) covering.

[9]Since E is locally path connected and connected, E is also path connected.

2.3 Covering Morphisms

We keep the conventions of section 2.2.

Definition 2.8 (Covering morphism)**.** Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be coverings of X. A *covering morphism* between p_1 and p_2 is map $\varphi : E_1 \to E_2$ such that $p_2 \circ \varphi = p_1$.

For a space X, we get a category $\mathbf{Cov}_X$ consisting of covering maps and the covering morphisms between them. A covering isomorphism is a covering morphism that is also a homeomorphism.

Proposition 2.7. *Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be two (connected) coverings.*

1. *If two covering morphims $\varphi_1 : E_1 \to E_2$ and $\varphi_2 : E_1 \to E_2$ agree at some point, then they are equal.*
2. *Fix $x \in X$, and let $\varphi : E_1 \to E_2$ be a covering morphism. Then φ restricts to an $\pi_1(X,x)$-equivariant map $p_1^{-1}(x) \to p_2^{-1}(x)$.*

Proof.

1. A morphism $\varphi : E_1 \to E_2$ is actually a lift of p_1 to p_2, so it follows from **Lemma 2.1** that if two covering morphisms agree at one point, that they are in fact equal.
2. Let $e \in E$, then $p_1(e) = x$ for some $x \in X$, and $p_2(\varphi(e)) = p_1(e) = x$, so $\varphi(e) \in p_2^{-1}(x)$, so φ restricts to a well-defined map from $p_1^{-1}(x)$ to $p_2^{-1}(x)$. Let $x \in X$, let $e \in p^{-1}(x)$, and let $[\alpha] \in \pi_1(X,x)$. We have to show that $\varphi([\alpha] \cdot e) = [\alpha] \cdot \varphi(e)$. By definition, $[\alpha] \cdot e = \tilde{\alpha}_e(1)$, where $\tilde{\alpha}_e$ is the lift of α in E_1 starting at e, so $\varphi([\alpha] \cdot e) = \varphi(\tilde{\alpha}_e(1))$. Since $p_2 \circ \varphi = p_1$, $\varphi \circ \tilde{\alpha}_e$ is the lift of α to E_2 starting at $(\varphi \circ \tilde{\alpha}_e)(0) = \varphi(e)$, so $[\alpha] \cdot \varphi(e) = (\varphi \circ \tilde{\alpha}_e)(1) = \varphi([\alpha] \cdot e)$, which proves the statement.

□

Let X be a space, and fix $x \in X$. From all the above, we can conclude that the map

$$\begin{aligned} F_x : \mathbf{Cov}_X &\longrightarrow \pi_1(X,x)\text{-}\mathbf{Sets} \\ (E,p) &\longmapsto p^{-1}(x) \\ (\varphi, (E_1,p_1), (E_2,p_2)) &\longmapsto \varphi : p_1^{-1}(x) \to p_2^{-1}(x) \end{aligned} \tag{33}$$

is a functor. It is called the *fibre functor*.

2.3.1 Covering automorphisms

All coverings are assumed to be connected, locally path connected and surjective.

Definition 2.9 (Automorphism group)**.** Let $p : E \to X$ be a covering. The automorphism group of E is the set $\mathrm{Aut}(E/X) := \{\varphi : E \to E : \varphi \text{ is a covering morphism}\}$

The group $\mathrm{Aut}(E/X)$ acts on E in a natural way via the action

$$\begin{aligned} \mathcal{A} : \mathrm{Aut}(E/X) \times E &\longrightarrow E \\ (\varphi, e) &\longmapsto \varphi(e). \end{aligned} \tag{34}$$

Definition 2.10 (Normalizer)**.** Let G be a group with a subgroup $H \subset G$. The normalizer of H in G is the subgroup $N_G(H) := \{g \in G : gHg^{-1} = H\}$, the biggest subgroup of G in which H is normal.

Proposition 2.8. *Let G be a group, and let S_1, S_2 be sets with transitive G-action. Let $s_1 \in S_1$ and $s_2 \in S_2$. There exists a G-equivariant bijection $\varphi : S_1 \to S_2$ such that $\varphi(s_1) = s_2$ if and only if* $\mathrm{Stab}_{s_1} = \mathrm{Stab}_{s_2}$.

Proposition 2.9. *The action of* $\mathrm{Aut}(E/X)$ *on E is free and permutes the components of $p^{-1}[U]$ for evenly covered $U \subset X$.*

Proposition 2.10. *Let $p_1 : E_1 \to X$ and $p_2 : E_2 \to X$ be two (connected) coverings, let $x \in X$, let $e_1 \in p_1^{-1}(x)$ and let $e_2 \in p_2^{-1}(x)$. There exists a (necessarily unique) covering isomorphism $\varphi : E_1 \to E_2$ such that $\varphi(e_1) = e_2$ if and only if $(p_1)_*[\pi_1(E_1,e_1)] = (p_2)_*[\pi_1(E_2,e_2)]$.*

Proof. Suppose there exists a covering isomorphism $\varphi : E_1 \to E_2$ such that $\varphi(e_1) = e_2$, then φ restricts to an equivariant map $p_1^{-1}(x) \to p_2^{-1}(x)$ by **Proposition 2.7**, which by **Proposition 2.8** and **Proposition 2.4** implies that $(p_1)_*[\pi_1(E_1, e_1)] = \mathrm{Stab}_{e_1} = \mathrm{Stab}_{e_2} = (p_2)_*[\pi_1(E_2, e_2)]$.

Suppose that $(p_1)_*[\pi_1(E, e_1)] = (p_2)_*[\pi_1(E, e_2)]$. By **Lemma 2.1** and **Theorem 2.3**, there exists covering morphism $\varphi : E_1 \to E_2$ and $\psi : E_2 \to E_1$ such that $\varphi(e_1) = e_2$ and $\psi(e_2) = e_1$. Then $\psi \circ \varphi$ and $\varphi \circ \psi$ agree with id_{E_1} on e_1 and id_{E_2} on e_2 respectively, so φ is a covering isomorphism. □

Proposition 2.11. *Let $p : E \to X$ be a (connected) covering. The action of* $\mathrm{Aut}(E/X)$ *on E is transitive on each fibre if and only if p is a normal (Galois) covering.*

Proof. Let $x \in X$ and $e_0, e_1 \in p^{-1}(x)$. By **Proposition 2.10** there exists a unique morphism $\varphi \in \mathrm{Aut}(E/X)$ such that $\varphi(e_0) = e_1$ if and only if $p_*[\pi_1(E, e_0)] = p_*[\pi_1(E, e_1)]$. This is true if and only if the conjugacy class of $p_*[\pi_1(E, e_0)]$ has only one element, which is the case if and only if $p_*[\pi_1(E, e_0)]$ is normal. □

Theorem 2.4. *Let $p : E \to X$, let $x \in X$ and let $e \in p^{-1}(x)$. Then*

$$\mathrm{Aut}(E/X) \cong \frac{N_{\pi_1(X,x)}(p_*[\pi_1(E, e)])}{p_*[\pi_1(E, e)]}. \tag{35}$$

Proof. Let $e_0, e_1 \in p^{-1}(x)$, and suppose there exists a unique morphism $\varphi \in \mathrm{Aut}(E/X$ such that $\varphi(e_0) = e_1$. By **Proposition 2.11**, we have $p_*[\pi_1(E, e_0)] = p_*[\pi_1(E, e_1)]$. Since E is connected the monodromy action is transitive, so there exists some $[\gamma] \in \pi_1(X, x)$ such that $\varphi(e_0) = [\gamma] \cdot e_0$. Let $[\beta] \in p_*[\pi_1(E, e_0)]$, then $[\beta] \cdot e_1 = e_1 \implies [\beta][\gamma] \cdot e_0 = [\gamma] \cdot e_0$, so $[\gamma]^{-1}[\beta][\gamma] \cdot e_0 = e_0 \implies [\beta] \in [\gamma]p_*[\pi_1(E, e_0)][\gamma]^{-1}$ and thus $[\gamma] \in N_{\pi_1(X,x)}[p_*[\pi_1(E, e_0)]]$.

Let $e_0 \in p^{-1}(x)$, let $[\gamma] \in N_{\pi_1(X,x)}[p_*[\pi_1(E, e_0)]]$, and define $e_1 := [\gamma] \cdot e_0$. It is clear that $e_1 \in p^{-1}(x)$. Let $[\beta] \in p_*[\pi_1(E, e_1)]$, then $[\gamma]^{-1}[\beta][\gamma] \cdot e_0 = e_0$, so $[\beta] \in [\gamma]p_*[\pi_1(E, e_0)][\gamma]^{-1} = p_*[\pi_1(E, e_0)]$. Let $[\beta] \in p_*[\pi_1(E, e_0)]$, then $[\gamma]^{-1}[\beta][\gamma] \cdot e_0 = e_0$, so $[\beta] \cdot e_1 = e_1$ and thus $[\beta] \in p_*[\pi_1(E, e_1)]$, so $p_*[\pi_1(E, e_0)] = p_*[\pi_1(E, e_1)]$. By **Proposition 2.10**, there exists a unique morphism $\varphi \in \mathrm{Aut}(E/X)$ such that $\varphi(e_0) = e_1 = [\gamma] \cdot e_0$. This defines a map

$$\begin{aligned} \Phi : N_{\pi_1(X,x)}(p_*[\pi_1(E, e_0)]) &\longrightarrow \mathrm{Aut}(E/X) \\ [\gamma] &\longmapsto \varphi_{[\gamma]} \end{aligned} \tag{36}$$

Let $[\gamma], [\gamma'] \in N_{\pi_1(X,x)}(p_*[\pi_1(E, e_0)])$. Then $\gamma\gamma'$ lifts to $\tilde{\gamma}_{e_0}\varphi_{[\gamma]} \circ \tilde{\gamma'}_{e_0}$, a path in E from e_0 to $(\varphi_{[\gamma]} \circ \varphi_{[\gamma']})(e_0)$. Hence $\varphi_{[\gamma\gamma']}(e_0) = [\gamma\gamma'] \cdot e_0 = (\varphi_{[\gamma]} \circ \varphi_{[\gamma']})(e_0)$, so $\varphi_{[\gamma\gamma']} = \varphi_{[\gamma]} \circ \varphi_{[\gamma']}$ (since they agree on one point and E is connected) and thus $\Phi([\gamma][\gamma']) = \Phi([\gamma]) \circ \Phi([\gamma])$. Since also $\Phi([c])(e) = [c] \cdot e = e$ for all $e \in E$, Φ is a group homomorphism.

Let $\varphi \in \mathrm{Aut}(E/X)$. Since $\pi_1(X, x)$ acts transitively, there exists some $[\gamma] \in \pi_1(X, x)$ such that $\varphi(e_0) = [\gamma] \cdot e_0$, and thus $\varphi = \varphi_{[\gamma]}$, so Φ is surjective.

Let $[\gamma] \in N_{\pi_1(X,x)}(p_*[\pi_1(E, e_0)])$, then $\varphi_{[\gamma]} \in \ker(\Phi) \iff \varphi_{[\gamma]} = \mathrm{id}_E \iff \varphi_{[\gamma]}(e_0) = e_0 = [\gamma] \cdot e_0 \iff [\gamma] \in p_*[\pi_1(E, e_0)]$, so $\ker(\Phi) = p_*[\pi_1(E, e_0)]$. □

Corollary 2.2. *Let $p : E \to X$ is a be a covering, let $x \in X$ and let $e \in p^{-1}(x)$.*

1. *If p is normal (Galois), then*

$$\mathrm{Aut}(E/X) \cong \frac{\pi_1(X, x)}{p_*[\pi_1(E, e)]}. \tag{37}$$

2. *If E is simply connected, then* $\mathrm{Aut}(E/X) \cong \pi_1(X, x)$.

Example 2.2. Consider the simply connected covering $\varepsilon : \mathbb{R} \to \mathbb{S}^1$ from **Example 2.1**. By **Corollary 2.2** we have $\mathbb{Z} = \pi_1(\mathbb{S}^1, 1) \cong \mathrm{Aut}(\mathbb{R}/\mathbb{S}^1)$. We can see this as follows. Let $f \in \mathrm{Aut}(\mathbb{R}/\mathbb{S}^1)$, then $e^{2\pi it} = e^{2\pi i f(t)}$ for all $t \in \mathbb{R}$, which is equivalent to saying that $f(t) - t \in \mathbb{Z}$ for all $t \in \mathbb{R}$. Hence the map sending $t \in \mathbb{R}$ to $f(t) - t \in \mathbb{R}$ is constant, so there exists some $m \in \mathbb{Z}$ such that $f(t) - t = m$ for all $t \in \mathbb{R}$, i.e. $f(t) = t + m$ for all $t \in \mathbb{R}$. The map $\beta : \mathbb{Z} \to \mathrm{Aut}(\mathbb{R}/\mathbb{S}^1)$ sending $m \in \mathbb{Z}$ to the map $f_m : \mathbb{R} \to \mathbb{R}$ sending $t \in \mathbb{R}$ to $t + m$ is then clearly a well-defined isomorphism, so indeed we find that $\mathrm{Aut}(\mathbb{R}/\mathbb{S}^1) \cong \mathbb{Z} = \pi_1(\mathbb{S}^1, 1)$.

Remark 2.3. If E_1 and E_2 are two simply connected coverings of X, then E_1 and E_2 are isomorphic as coverings.

Definition 2.11 (Universal covering)**.** Let X be a space. Any simply connected covering $\tilde{X}$ of X is called a *universal covering*, and is unique up to isomorphism.

Question 2.2. *When does a space X have a universal covering?*

Definition 2.12 (Semilocally simply connected)**.** A space X is semilocally simply connected if every point in X has a neighborhood N_x such that every loop at x in N_x is homotopic in X to a constant loop.

In contrast, a space is locally simply connected if every neighborhood of every point contains a simply connected neighborhood.

Theorem 2.5. *A connected and locally path connected space X has a universal covering if and only if X is semilocally simply connected.*

A universal covering $u : \tilde{X} \to X$ is called universal because it dominates any other covering, i.e. if $p : E \to X$ is another covering, then there exists a covering map $q : \tilde{X} \to E$ such that $p \circ q = u$.

2.3.2 Group actions on topological spaces

Let E be a connected, locally path connected space. Suppose a group Γ acts on E. We get a quotient map $p_\Gamma : E \to E/\Gamma$.

Definition 2.13 (Covering space action)**.** The action of Γ on E is called a *covering space action* if and only if the action is faithful/effective and for all $e \in E$ there exists a neighborhood $U_e \subset E$ such that $\gamma U_e \cap U_e = \emptyset$ for all $\gamma \in \Gamma \setminus \{e_\Gamma\}$

Proposition 2.12. *In this situation, $p_\Gamma : E \to E/\Gamma$ is a covering map if and only if the action is a covering space action, in which case it is a normal (Galois) covering and* $\mathrm{Aut}({}^{E}/_{E/\Gamma}) = \Gamma$.

Proof. For all $\gamma \in \Gamma$, the map $m_\gamma : E \to E$ sending $e \in E$ to $\gamma \cdot e$ is an element of $\mathrm{Aut}({}^{E}/_{E/\Gamma})$. Since the action is faithful, we have the inclusion $\Gamma \subset \mathrm{Aut}({}^{E}/_{E/\Gamma})$. By definition, Γ acts transitively on the fibres of p_Γ. Let $\varphi \in \mathrm{Aut}({}^{E}/_{E/\Gamma})$, and let $e, e' \in E$ such that $\varphi(e) = e'$. There exists an element $\gamma \in \Gamma$ such that $\gamma \cdot e = e'$, so φ and m_γ agree at one point and thus $\varphi = m_\gamma$, so $\varphi \in \Gamma$ and thus $\Gamma = \mathrm{Aut}({}^{E}/_{E/\Gamma})$. Now $\mathrm{Aut}({}^{E}/_{E/\Gamma}$ acts transitively on the fibres of p_Γ, which by **Proposition 2.11** implies that p_Γ is normal. □

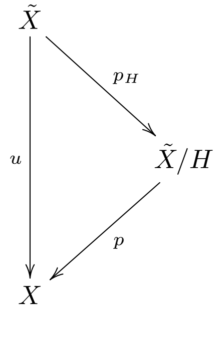
Definition 2.14 (Galois covering)**.** In this situation, $p_\Gamma : E \to E/\Gamma$ is called a *Galois covering* with Galois group Γ.

2.4 Galois Theory of Coverings I

Let X be a connected, locally path connected space that has a universal covering $u : \tilde{X} \to X$. Fix $x \in X$ and $\tilde{x} \in u^{-1}(x)$.

Let $p : E \to X$ be a connected covering, and let $e \in p^{-1}(x)$. Then $p_* : \pi_1(E, e) \to \pi_1(X, x)$ is injective, so $p_*[\pi_1(E, e)]$ is a subgroup of $\pi_1(X, x)$.

Let $H \subset \pi_1(X, x)$ be a subgroup of $\pi_1(X, x)$. Recall that $\mathrm{Aut}(\tilde{X}/X) = \pi_1(X, x)$, so H can be viewed as a subgroup of $\mathrm{Aut}(\tilde{X}, X)$. As such, H acts on $\tilde{X}$ via covering automorphims. This is a covering space action, so $p_H : \tilde{X} \to \tilde{X}/H$ is a covering map. Since[10] u is H-invariant, we get a commutative diagram



Proposition 2.13.

1. *The map $p : \tilde{X}/H \to X$ is a covering map.*

[10]Let $\varphi \in \mathrm{Aut}(\tilde{X}/X)$ and let $\tilde{x} \in \tilde{X}$, then $u(\varphi(\tilde{x}) = u(\tilde{x}))$, so u is H-invariant.

2. $p_*[\pi_1(\tilde{X}/H, p_H(\tilde{x}))] = H$.

Proof.

1. Let $x \in X$, let U be an evenly covered (by u) neighborhood of x, and let U_0^H be a component of $p^{-1}[U]$. Since $\tilde{X}/H$ is locally path connected, U_0^H open and closed in $p^{-1}[U]$, so $p_H^{-1}[U_0^H]$ is open and closed in $p_H^{-1}[p^{-1}[U]] = u^{-1}[U]$, which implies that $p_H^{-1}[U_0^H]$ is a union of components of $u^{-1}[U]$. Let $U_0 \subset u^{-1}[U]$ a component, then $u|_{U_0} = p|_{U_0^H} \circ p_H|_{U_0}$, so $p_H|_{U_0}$ is injective. Since p_H is an open and surjective map which is H-invariant, $p_H|_{U_0}$ is a homeomorphism, so $p|_{U_0^H} = u|_{U_0} \circ (p_H|_{U_0})^{-1}$ is also a homeomorphism as a composition of homeomorphisms, so p is a covering map.

2. Note that $p_*[\pi_1(\tilde{X}/H, p_H(\tilde{x}))] = \mathrm{Stab}_{p_H(\tilde{x})}$ under the monodromy action of $\pi_1(X, x)$, by **Proposition 2.4**. Since p_H restricts to an $\pi_1(X, x)$-invariant map from $u^{-1}(x)$ to $p^{-1}(x)$, we have $[\gamma] \cdot p_H(\tilde{x}) = p_H([\gamma] \cdot \tilde{x})$ for all $[\gamma] \in \pi_1(X, x)$. Then $[\gamma] \in \mathrm{Stab}_{p_H(\tilde{x})}$ if and only if $p_H([\gamma] \cdot \tilde{x}) = p_H(\tilde{x})$, i.e. if and only if there exists some element $h \in H$ such that $[\gamma] \cdot \tilde{x} = h \cdot \tilde{x}$. Since $\tilde{X}$ is simply connected, the monodromy action is free, so $[\gamma] = h \in H$, and thus $\mathrm{Stab}_{p_H(\tilde{x})} = H = p_*[\pi_1(\tilde{X}/H, p_H(\tilde{x}))]$.

□

Fix a space X that has a universal covering $u : \tilde{X} \to X$. Let $x \in X$, and let $\tilde{x} \in u^{-1}(x)$.

Theorem 2.6 (Galois correspondence for pointed coverings)**.** *The assignments*

$$\begin{aligned} \textbf{P-Cov}_X &\longrightarrow \{\text{Subgroups of } \pi_1(X, x)\} \\ \begin{pmatrix} p : E \to X \\ e \in p^{-1}(x) \end{pmatrix} &\longmapsto p_*[\pi_1(E, e)] \\ \begin{pmatrix} p : \tilde{X}/H \to X \\ \tilde{x} \bmod H \end{pmatrix} &\longleftarrow H \end{aligned} \tag{38}$$

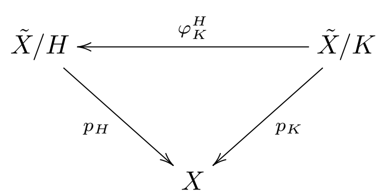
are inverse to each other. This gives a one-to-one correspondence between pointed coverings of X and subgroups of $\pi_1(X, x)$.

Remark 2.4. If we forget the basepoint, then we work with conjugacy classes and isomorphism classes of coverings.

Proposition 2.14. *Let $H \subset \pi_1(X, x)$ be a subgroup.*

1. *The covering $p : \tilde{X}/H \to X$ is Galois (normal) if and only if $H \triangleleft \pi_1(X, x)$.*

2. *If $K \subset H \subset \pi_1(X, x)$, then*



commutes and φ_K^H is a covering morphism.

Proof.

1. If $H \triangleleft \pi_1(X, x)$, then $\pi_1(X, x)/H$ is a group. Then $p : \tilde{X}/H \to X$ is the quotient by $\pi_1(X, x)/H$, so by **Proposition 2.12**, $p : \tilde{X}/H \to X$ is Galois with Galois group $\pi_1(X, x)/H$. If $p : \tilde{X}/H \to X$ is normal then (by **Proposition 2.13**) $p_*[\pi_1(\tilde{X}/H, p_H(\tilde{x}))] = H$ is normal.

2. By **Proposition 2.13**, p_H and p_K are both covering maps, and the diagram clearly commutes.

□

Galois correspondences		
Universal objects:	Coverings of X	Field extensions of K
Galois groups:	$\pi_1(X,x) = \mathrm{Aut}(\tilde{X}/X)$	$\mathrm{Gal}(\overline{K}/K) = \mathrm{Aut}(\overline{K}/K)$
Correspondences:	{Subgroups of $\pi_1(X,x)$} $\updownarrow$ {Pointed coverings of X}	{(Closed) Subgroups of $\mathrm{Gal}(\overline{K}/K)$} $\updownarrow$ {Field extensions L/K}
Galois objects:	Normal subgroups of $\pi_1(X,x)$	Normal subgroups of $\mathrm{Gal}(\overline{K}/K)$

2.5 Galois Theory of Covering Maps II

All coverings are surjective and locally path connected, but not necessarily connected.

2.5.1 G-Coverings and G-sets

Let G be a group and $p : Y \to X$ be a G-covering, i.e. there exists a covering space action of G on Y and $p : Y \to Y/G = X$ is the quotient map. Let T be a G-set. Consider the diagonal action of G on $Y \times T$ defined by $g \cdot (y,t) = (g \cdot y, g \cdot t)$ for all $g \in G$ and for all $(y,t) \in Y \times T$. Give T the discrete topology, and define $Y_T := (Y \times T)/G$. Let q_T^Y be the quotient map. The map

$$\begin{aligned} P : Y \times T &\longrightarrow X \\ (y,t) &\longmapsto p(y) \end{aligned} \tag{39}$$

is G-invariant, i.e. for all $g \in G$ we have $P((g \cdot y, g \cdot t)) = p(g \cdot y) = p(y)$. Hence we get a map $p_T : Y_T \to X$ such that

$$\begin{array}{ccc} Y \times T & \xrightarrow{\;P\;} & X \\ {\scriptstyle q_T^Y}\big\downarrow & \nearrow{\scriptstyle p_T} & \\ Y_T & & \end{array}$$

commutes, sending $\overline{(y,t)} \in Y_T$ to $p(y) \in X$. This map is a covering map.

Let $\varphi : T \to T'$ be an G-equivariant map of G-sets. Then the map $q_{T'}^Y \circ \mathrm{id}_Y \times \varphi$ is G-invariant, since for all $(y,t) \in Y \times T$ and all $g \in G$ we have

$$\begin{aligned} (q_{T'}^Y \circ \mathrm{id}_Y \times \varphi)((y,t)) &= q_{T'}^Y((y,\varphi(t))) \\ &= \overline{(y,\varphi(t))} \\ &= \overline{(g \cdot y, g \cdot \varphi(t))} \\ &= q_{T'}^Y((g \cdot y, g \cdot \varphi(t))) \\ &= q_{T'}^Y((g \cdot y, \varphi(g \cdot t))) \\ &= q_{T'}^Y(\mathrm{id}_Y \times \varphi((g \cdot y, g \cdot t))) \\ &= (q_{T'}^Y \circ \mathrm{id}_Y \times \varphi)((g \cdot y, g \cdot t)), \end{aligned} \tag{40}$$

so we get a commutative diagram

$$\begin{array}{ccc} Y_T & \xrightarrow{\;f_\varphi\;} & Y_{T'} \\ & {\scriptstyle p_T}\searrow \quad \swarrow{\scriptstyle p_{T'}} & \\ & X & \end{array}$$

where f_φ is the map sending $\overline{(y,t)}$ to $\overline{(y,\varphi(t)}$.

Remark 2.5. Suppose that T is a transitive G-set, and let $t_0 \in T$. We have an iso of G-sets $G/H \cong T$ sending gH to $g \cdot t_0$, where $H = \mathrm{Stab}_{t_0}$ (note that H is not necessarily normal).

Proposition 2.15. *In this situation, $Y_T \simeq Y/H$ as coverings of X. In particular, if Y is connected, so is $Y_T = Y_{G/H}$.*

Proof. Since $G/H \cong T$ (as G-sets), we can prove the proposition for $T = G/H$. Define the map

$$\begin{aligned} \mathcal{H} : Y \times T &\longrightarrow Y/H \\ (y, gH) &\longmapsto g^{-1} \cdot y \bmod H. \end{aligned} \tag{41}$$

Then for all $\bar{g} \in G$ we have $\mathcal{H}((\bar{g} \cdot y, \bar{g}gH)) = g^{-1}\bar{g}^{-1}\bar{g} \cdot y \bmod H = g^{-1} \cdot y \bmod H$, so $\mathcal{H}$ is G-invariant and we get a map $\tilde{\mathcal{H}} : Y_T \to Y/H$. The map[11]

$$\begin{aligned} \mathcal{I} : Y &\longrightarrow Y_T \\ y &\longmapsto \overline{(y, eH)} \end{aligned} \tag{42}$$

is clearly H-invariant, so we get a map $\tilde{\mathcal{I}} : Y/H \to Y_T$. Then $\tilde{\mathcal{H}} \circ \tilde{\mathcal{I}} = \mathrm{id}_{Y/H}$, $\tilde{\mathcal{I}} \circ \tilde{\mathcal{H}} = \mathrm{id}_{Y_T}$ and[12] $p_H \circ \tilde{\mathcal{H}} = \tilde{\mathcal{I}}$, so $Y_T \simeq Y/H$ as coverings of X. □

Remark 2.6. Let T be any G-set. Then we can write $T = \bigsqcup_{\alpha \in L} T_\alpha$ as a disjoint union of orbits for some indexing set L, and $Y_T = \bigsqcup_{\alpha \in L} Y_{T_\alpha}$ as coverings of X. Since each T_α is transitive, we have an isomorphism of G-sets $T_\alpha \simeq G/H_\alpha$ for each $\alpha \in L$, and thus $Y_T \simeq \bigsqcup_{\alpha \in L} Y/H_\alpha$ as coverings of X.

Now suppose X is a connected, locally path connected space that has a universal covering $u : \tilde{X} \to X$. Let $x \in X$ and $\tilde{x} \in u^{-1}(x)$. Then u is a $\pi_1(X, x)$-covering $\tilde{X} \to \tilde{X}/\pi_1(X, x) = X$.

2.5.2 Galois Theory for general (non-connected) coverings

The constructions of section 2.5.1 define a functor

$$\begin{aligned} \pi_1(X, x)\text{-}\mathbf{Sets} &\longrightarrow \mathbf{Cov}_X \\ T &\longmapsto \tilde{X}_T \\ (\varphi : T \to T') &\longmapsto (f_\varphi : \tilde{X}_T \to \tilde{X}_{T'}) \end{aligned} \tag{43}$$

Proposition 2.16. *For all $\pi_1(X, x)$-sets T, the fibre $p_T^{-1}(x)$ of $p_T : \tilde{X}_T \to X$ given by $p_T(\overline{(z, t)}) = u(z)$ over x equals T.*

Proof. The assignments

$$\begin{aligned} T &\longrightarrow p_T^{-1}(x) \\ t &\longmapsto \overline{(\tilde{x}, t)} \\ [\gamma]^{-1} \cdot t &\longleftarrow \overline{(z, t)} = \overline{([\gamma] \cdot \tilde{x}, t)} \end{aligned} \tag{44}$$

are inverse to each other. □

Recall that we had a functor $F_x : \mathbf{Cov}_X \to \pi_1(X, x)\text{-}\mathbf{Sets}$ which send a covering (E, p) to $p^{-1}(x)$, the fibre functor.

Proposition 2.17. *Let $p : E \to X$ be a covering. Then $\tilde{X}_{p^{-1}(x)} \simeq E$.*

Proof. Write E as a disjoint union of connected components, i.e.

$$E = \bigsqcup_{\alpha \in L} E_\alpha, \tag{45}$$

and let

$$\bigsqcup_{\alpha \in L} p_\alpha : \bigsqcup_{\alpha \in L} E_\alpha \to X, \tag{46}$$

[11] Here $e \in G$ is the identity element.

[12] Here $p_H : Y/H \to X$ is the restriction of the quotient map $p : Y \to X$.

then the fibre of x is

$$p^{-1}(x) = \bigsqcup_{\alpha \in L} p_\alpha^{-1}(x). \tag{47}$$

Since E_α is connected for each $\alpha \in L$, $p_\alpha^{-1}(x)$ is a transitive $\pi_1(X,x)$-set. Let $e_\alpha \in p_\alpha^{-1}(x)$. Then (by Remark 2.5) we have

$$\begin{aligned} p_\alpha^{-1}(x) &= \pi_1(X,x)/(p_\alpha)_*[\pi_1(E_\alpha, e_\alpha)] \\ &= \pi_1(X,x)/H_\alpha. \end{aligned} \tag{48}$$

Since (by **Proposition 2.3**)

$$\begin{aligned} \tilde{X}_{p_\alpha^{-1}(x)} &\simeq \tilde{X}/H_\alpha \\ &\simeq E_\alpha, \end{aligned} \tag{49}$$

we find that

$$\begin{aligned} \tilde{X}_{p^{-1}(x)} &= \bigsqcup_{\alpha \in L} \tilde{X}_{p_\alpha^{-1}(x)} \tilde{X}_{p_\alpha^{-1}(x)} \\ &= \bigsqcup_{\alpha \in L} \tilde{X}/H_\alpha \\ &\simeq \bigsqcup_{\alpha \in L} E_\alpha \\ &= E. \end{aligned} \tag{50}$$

□

We can conclude:

The fibre functor

$$F_x : \mathbf{Cov}_X \longmapsto \pi_1(X,x)\text{-}\mathbf{Sets} \tag{51}$$

is an equivalence of categories, i.e there exists a functor

$$G : \pi_1(X,x)\text{-}\mathbf{Sets} \longrightarrow \mathbf{Cov}_X \tag{52}$$

such that

$$\begin{aligned} F_x \circ G &\simeq \mathrm{id}_{\pi_1(X,x)\text{-}\mathbf{Sets}} \\ G \circ F_x &\simeq \mathrm{id}_{\mathbf{Cov}_X} \end{aligned} \tag{53}$$

Corollary 2.3. *If G is any group, we have a bijection*

$$\mathrm{Hom}_{\mathbf{Grp}}(\pi_1(X,x), G) \xrightarrow{\sim} \{\text{Isomorphism classes of } G\text{-Coverings of } X\} \tag{54}$$

2.5.3 Proof of the Seifert-Van Kampen Theorem

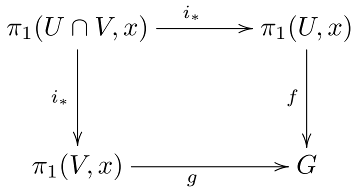
Theorem 2.7 (Seifert-Van Kampen)**.** *Let X be a space, and let U and V be open subsets of X such that $X = U \cup V$. Suppose that X, U, V and $U \cap V$ have universal coverings, and let $x \in U \cap V$. Then $\pi_1(X,x)$ is the pushout of*

$$\begin{array}{ccc} \pi_1(U \cap V, x) & \xrightarrow{i_*} & \pi_1(U,x) \\ \downarrow{\scriptstyle i_*} & & \\ \pi_1(V,x) & & \end{array}$$

in **Grp**. *Hence*

$$\pi_1(X,w) = \pi_1(U,w) \underset{\pi_1(U\cap V,w)}{\star} \pi_1(V,w). \tag{55}$$

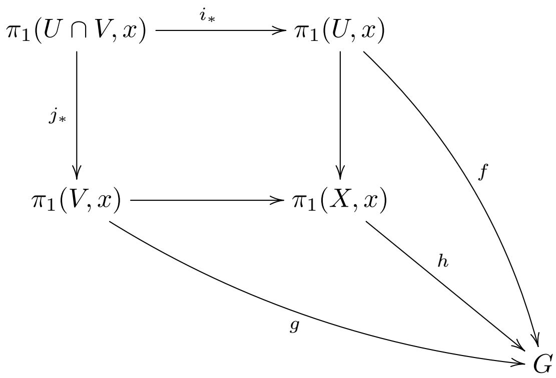
Proof. Let G be any group and let f, g be maps such that the diagram



commutes. By **Corollary 2.3**, the maps f and g correspond to G-coverings $p_f : E_f \to U$ and $p_g : E_g \to V$. The fact that $f \circ i_* = g \circ j_*$ means that there exists an isomorphism of coverings $p_g^{-1}(U \cap V) \underset{\varphi}{\xrightarrow{\sim}} p_f^{-1}(U \cap V$, so we can glue. Set

$$E = \frac{E_f \sqcup E_g}{\forall\, x : x \sim \varphi(x)} \xrightarrow{p} X, \tag{56}$$

then this a G-covering of X, so we get a map $\pi_1(X, x) \xrightarrow{h} G$. Since (as G-coverings) $p^{-1}(U) \simeq E_f$ and $p^{-1}(V) \simeq E_g$, the diagram



commutes, so $\pi_1(X, x)$ is the pushout. $\square$

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