Tautological differential forms on moduli spaces of curves

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Contents

Introduction

Background

Let $g \geq 2$. Mumford [Mum83] set up a Chow ring on the moduli space $\mathcal{M}_g$ of genus g curves (and on its Deligne–Mumford compactification $\overline{\mathcal{M}_g}$). He modeled this construction after the enumerative geometry of the Grassmannians, where canonical elements of the Chow ring of a Grassmannian are given by the Chern classes of the universal vector bundle and the universal quotient bundle. He defined canonical classes κ_i $(i \geq 0)$ in the Chow ring of $\mathcal{M}_g$ by taking pushforwards of powers of the first Chern class of the relative cotangent bundle $\omega_{\mathcal{C}_g/\mathcal{M}_g}$ of the universal family $p : \mathcal{C}_g \to \mathcal{M}_g$. The *tautological ring* $R^*(\mathcal{M}_g)$ is then defined as the subring of the Chow ring of $\mathcal{M}_g$ generated by these κ-classes. Mumford proved that all Chern classes of the Hodge bundle $p_*\omega_{\mathcal{C}_g/\mathcal{M}_g}$ of the universal family lie in the tautological ring $\mathcal{M}_g$.

Moreover, Mumford proved that the tautological ring $R^*(\mathcal{M}_g)$ is generated by the classes $\kappa_1, \ldots, \kappa_{g-2}$. Looijenga [Loo95] then proved that the tautological ring $R^*(\mathcal{M}_g)$ vanishes in degrees higher than $g-2$, and that $R^{g-2}(\mathcal{M}_g)$ is at most one-dimensional, and spanned by κ_{g-2}. Faber [Fab97] then proved that κ_{g-2} is nonzero, and hence that $R^{g-2}(\mathcal{M}_g)$ is one-dimensional. He then conjectured in [Fab99] that $R^*(\mathcal{M}_g)$ is a *Gorenstein algebra*: that is, for all $0 \leq d \leq g-2$ the pairing

$$R^d(\mathcal{M}_g) \times R^{g-2-d}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q},$$

induced by multiplication in the Chow ring, is perfect. Faber's conjecture has been verified for all $g \leq 23$ (see [Fab13]), but for $g = 24$ not enough relations have yet been found to verify the conjecture.

Rather than fixing $g \geq 2$, one could study the behavior of the cohomology of $\mathcal{M}_g$ as g tends to infinity. Mumford conjectured in [Mum83] that the homomorphism $\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \to H^*(\mathcal{M}_g, \mathbb{Q})$ is an isomorphism in degrees $\leq k$, where k tends to infinity as g tends to infinity. In other words: the only cohomology classes that occur on $\mathcal{M}_g$ for all $g \geq 2$ are the κ-classes, and there are no nontrivial relations among these classes that hold for arbitrary values of g. Harer [Har85] showed that for all $k \geq 0$ there is a g for which we have isomorphisms

$$H^k(\mathcal{M}_g; \mathbb{Q}) \cong H^k(\mathcal{M}_{g+1}; \mathbb{Q}) \cong H^k(\mathcal{M}_{g+2}; \mathbb{Q}) \cong \ldots.$$

In other words: for each $k \geq 0$ the kth cohomology of $\mathcal{M}_g$ stabilizes as g tends to infinity. Miller [Mil86] and Morita [Mor87] showed that the homomorphism

$$\mathbb{Q}[\kappa_1, \kappa_2, \ldots] \to H^*(\mathcal{M}_g; \mathbb{Q})$$

is *injective* in degrees $\leq k$, where k tends to infinity as g tends to infinity. Finally, Mumford's conjecture was proved by Madsen and Weiss in [MW07].

The moduli space $\mathcal{M}_g$ has also been studied using analytical tools. Faltings [Fal84] assigns to each admissible line bundle L on a Riemann surface X a hermitian metric on the determinant of cohomology $\det R\Gamma(L)$. By comparing these metrics with a canonical metric on the line bundle $O(-\Theta)$ on the Jacobian, Faltings obtains an invariant $\delta(X)$ of the Riemann surface X. This invariant gives rise to a function $\delta_g : \mathcal{M}_g \to \mathbb{R}$. Hain and Reed [HR04] construct a natural metric on the biextension line bundle on $\mathcal{M}_g$. This line bundle is isomorphic to the $(8g+4)$th power of the Hodge bundle; by comparing metrics they obtain another invariant $\beta_g : \mathcal{M}_g \to \mathbb{R}$, defined up to a constant. Another invariant $a_g = 2\pi\varphi_g : \mathcal{M}_g \to \mathbb{R}$ was found by Kawazumi [Kaw08; Kaw09] and Zhang [Zha10] in different contexts. Kawazumi constructed two differential 2-forms e^J, e^A on $\mathcal{C}_g$ that both represent the class of the relative tangent bundle of the universal family $\mathcal{C}_g \to \mathcal{M}_g$. He shows that these forms are not equal but are related by the identity

$$e^A - e^J = \frac{-2\sqrt{-1}}{2g(2g+1)} \partial\bar{\partial} a_g.$$

De Jong [dJon13; dJon16] showed that the invariants of Faltings, Hain–Reed, and Kawazumi–Zhang are linearly dependent.

Overview

In this thesis we will generate differential 2-forms on the moduli space $\mathcal{M}_g$, the universal family $\mathcal{C}_g \to \mathcal{M}_g$, and higher powers of $\mathcal{C}_g$, by listing various canonical hermitian line bundles on these moduli spaces and taking their first Chern forms. By using these 2-forms we will construct an analytic analogue to the tautological rings, the rings of *tautological differential forms*. We will show that these rings are not 'too big' (i.e. they are finite-dimensional in each degree, and hence finite-dimensional). By using a canonical line bundle on the universal Jacobian bundle $\mathcal{J}_g \to \mathcal{M}_g$ we will moreover be able to compute various relations between tautological differential forms. We will carry out some of these computations in a combinatorial framework based on marked graphs.

In Chapter 1 we will recall the general theory of families of manifolds. We will show that every family of oriented manifolds admits a uniquely defined fiber integral operator. Afterwards we will specialize to holomorphic families of compact Riemann surfaces of a fixed genus. Some hermitian line bundles that appear canonically on such families will be constructed, and we will find some canonical isometries among them.

The theory of moduli spaces is explored in Chapter 2. We will recall that no fine moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus g exists in the category of complex manifolds. The problem is that compact Riemann surfaces admit nontrivial automorphisms that can be exploited to construct nontrivial isotrivial families by twisting. One solution to this problem is adding extra structure to our Riemann surfaces, such as Teichmüller structures, thereby annihilating any nontrivial automorphisms. Another solution, by Deligne and Mumford [DM69], is to construct the moduli space $\mathcal{M}_g$ as a stack, rather than a complex manifold. Although $\mathcal{M}_g$ is not a complex manifold anymore, we may still define objects such as differential forms and hermitian line bundles on $\mathcal{M}_g$, and we will discuss why such objects on $\mathcal{M}_g$ can be viewed as objects that occur universally on families of compact Riemann surfaces of genus g.

In Chapter 3 we will take a detour and discuss r-marked graphs, which are graphs for which r vertices are marked with the integers $1, \ldots, r$. These graphs give us a combinatorial framework for working with tautological differential forms in Chapter 4. We will discuss contraction operations on marked graphs, and show that there are only finitely many contracted graphs up to isomorphism. In fact, given an integer $d \in \mathbb{Z}$, the number of isomorphism classes of contracted r-marked graphs of characteristic $r - d$ can be expressed as a polynomial in r:

Theorem 3.8.1. Let $d \in \mathbb{Z}$ be an integer. If d is negative, then for any $r \geq 0$ there are no contracted r-marked graphs of characteristic $r - d$.
If $d \geq 0$, then there exists a polynomial $f_d \in \mathbb{Q}[x]$ of degree $2d$ such that the number of isomorphism classes of contracted r-marked graphs of characteristic $r - d$ is equal to $f_d(r)$ for all $r \geq 0$. The leading coefficient of f_d is $1/(2^d \cdot d!)$.

We will compute the polynomial f_d for all $d \leq 4$.

Finally, in Chapter 4, we will construct rings of tautological differential forms on $\mathcal{M}_g$, $\mathcal{C}_g$, and $\mathcal{C}_g^r$ ($r \geq 2$), where $\mathcal{C}_g \to \mathcal{M}_g$ is the universal family of compact Riemann surfaces of genus g. Not every definition of the tautological Chow rings translates immediately to the analytical setting; we will discuss several of these definitions and see which translates best. Next we will give a method of constructing tautological differential forms from marked graphs, and show that tautological forms associated to contracted graphs span the ring of tautological forms, thereby proving that the rings of tautological forms are finite-dimensional in each degree, and hence finite-dimensional:

Theorem 4.6.1. For all $r \geq 0$ and $g \geq 2$, the ring of tautological forms $\mathcal{R}^*(\mathcal{C}_g^r)$ is finite-dimensional.

We will fully compute the degree 2 part of the ring of tautological forms on $\mathcal{C}_g^r$ for all $r \geq 0$, and provide an algorithm for computing more relations among tautological differential forms associated to marked graphs.

Chapter 1

Families of curves

The first chapter serves as a preliminary chapter, where we establish some of the notation and theory we will use in Chapter 4. In Section 1.1 we will study families of manifolds, which are roughly speaking collections of manifolds smoothly parametrized by another manifold. Moreover we show why the category of manifolds does not have all fiber products, and show that fiber products of families of manifolds do exist. In Section 1.2 we will study currents, and in particular we will look at the pushforward operator on currents. Afterwards, in Section 1.3, we will be constructing the fiber integral operator along families of manifolds, which is the smooth analogue of the pushforward operator on currents. Finally, in Section 1.4 we will discuss hermitian vector bundles, and construct some hermitian line bundles that appear canonically on families of curves. The theory discussed in this section will allow us to construct tautological forms on families of curves and find relations amongst them, which will be done in Chapter 4.

In this thesis the terms ‘manifold’ and ‘smooth manifold’ mean the same thing: a locally ringed space that is second-countable, Hausdorff, and locally isomorphic to the Euclidean space $\mathbb{R}^n$ (for some $n \geq 0$) with its sheaf of smooth real-valued functions. Equivalently, a manifold is a second-countable Hausdorff topological space equipped with a smooth atlas. While we do not require that manifolds are connected, we do assume that all manifolds are equidimensional. This is merely for our convenience; most theory immediately generalizes to nonequidimensional manifolds by reducing to equidimensional components.

Likewise, complex manifolds are always assumed to be second-countable, Hausdorff, and equidimensional.

We will assume that the reader is familiar with the elementary theory of manifolds as in [Lee03] and [dRha84]. For the reader’s convenience we will be repeating some definitions.

1.1 Families of manifolds

In this section, we consider submersions: morphisms of manifolds whose fibers are, again, manifolds. Moreover, we will define oriented submersions. These are submersions whose fibers are equipped with orientations that vary continuously.

1.1.1 Submersions

Let $f : X \to Y$ be a morphism of manifolds. Recall that f is a *submersion* if for all points $x \in X$ the associated map of tangent spaces $df_x : T_{X,x} \to T_{Y,f(x)}$ is surjective. So submersions are the analytic analogue to the smooth morphisms in the setting of algebraic geometry.

> **Example 1.1.1.** If X and Y are two manifolds, then the projection $p_2 : X \times Y \to Y$ is a submersion.

By the Constant Rank Theorem ([Lee03, Theorem 4.12]) a submersion locally looks like a projection $\mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^n$. More precisely, if $f : X \to Y$ is a submersion, then for each point $x \in X$ we can construct a commutative diagram

$$\begin{tikzcd}
U \arrow[r, hook] \arrow[d, "f|_U"] & \mathbb{R}^r \times \mathbb{R}^n \arrow[d, "p_2"] \\
V \arrow[r, hook] & \mathbb{R}^n
\end{tikzcd}$$

where $n = \dim(Y)$ and $r + n = \dim(X)$, where $U \subseteq X$ and $V \subseteq Y$ are open neighborhoods of x and $f(x)$, respectively, such that $f(U) \subseteq V$, and where the horizontal arrows are open immersions of manifolds.

Note that in the above situation the fibers of f are locally isomorphic to $\mathbb{R}^r$. Indeed, the fibers of a submersion $f : X \to Y$ are properly embedded submanifolds of X; see [Lee03, Theorem 5.12]. In other words, the fibers of a submersion $f : X \to Y$ form a family of manifolds parametrized by the points of Y. By using [Lee03, Theorem 5.29] one can easily show that for any $y \in Y$ the fiber $X_y = f^{-1}(y)$ is the fiber product in the category of manifolds of the morphisms $\{y\} \to Y$ and $f : X \to Y$:

$$X_y = X \times_Y \{y\}.$$

In Section 1.1.2 we will look further into fiber products in the category of manifolds.

> **Definition 1.1.2.** A *family of manifolds* is a surjective submersion of manifolds $f : X \to Y$. A *family of compact/connected/... manifolds* is a family of manifolds whose fibers are all compact/connected/... manifolds.

> **Lemma 1.1.3.** Let $f : X \to Y$ be a family of compact connected manifolds. Then f is a proper map.

Proof. First let $f : X \to Y$ be any continuous map of topological spaces. We define an equivalence relation $\sim$ on X, where two points x_1, x_2 are equivalent if and only if $f(x_1) = f(x_2) =: y$ and the points x_1, x_2 lie in the same connected component of the fiber $f^{-1}(y)$. The *component decomposition* is the decomposition of f into the continuous maps $X \to X/\sim \to Y$. By [WD79, B.III] the quotient map $X \to X/\sim$ is closed if X and Y are Hausdorff, X is locally compact, and all connected components of all fibers of f are compact.

This is clearly the case if $f : X \to Y$ is a family of connected compact manifolds. Moreover, in that case, the map $X/\sim \to Y$ is a bijection. As f is a surjective submersion, it is itself a quotient map ([Lee03, Proposition 4.28]). It follows that the map $X/\sim \to Y$ is a homeomorphism, and hence f is closed.

We find that f is a closed map with compact fibers, and therefore f is proper ([Lee03, A.53]). □

Ehresmann's fibration theorem [Ehr52; Voi02, Theorem 9.3] states that proper submersions with a contractible base are, in fact, trivial fiber bundles.

Theorem 1.1.4 (Ehresmann)**.** Let $f : X \to Y$ be a proper submersion of manifolds, and assume that Y is contractible. Then f is a trivial smooth fiber bundle. □

As every manifold can be covered with contractible opens, we immediately obtain the following.

Corollary 1.1.5. Let $f : X \to Y$ be a proper submersion of manifolds. Then f is a smooth fiber bundle. □

In particular we find that families of compact connected manifolds are smooth fiber bundles.

We will be using the following lemma later.

Lemma 1.1.6. Let $f : X \to Y$ be a submersion of manifolds. Then the pullback operator on differential forms

$$f^* : A^*(Y) \to A^*(X)$$

is injective.

Proof. Let $x \in X$ be a point. As f is a submersion, the tangent map $df : T_{X,x} \to T_{Y,f(x)}$ is surjective, and dually, the cotangent map $T^*_{Y,f(x)} \to T^*_{X,x}$ is injective. Taking exterior algebras yields the pullback map

$$\bigwedge T^*_{Y,f(x)} \to \bigwedge T^*_{X,x},$$

which is injective, too. As differential forms on Y and X are sections of the bundles $\bigwedge T^*_Y$ and $\bigwedge T^*_X$, respectively, the lemma follows. □

1.1.2 Fiber products of manifolds

The category **Man** of manifolds is not as well-behaved as, say, the category of schemes. For instance, the category **Man** does not have all fiber products. In this section we will show that a fiber product of two morphisms of manifolds does exist if one of the morphisms is a submersion.

Proposition 1.1.7. Let $f : X \to S$ and $g : Y \to S$ be morphisms of manifolds. If f is a submersion, then the fiber product $X \times_S Y$ exists in **Man**. The underlying topological space of $X \times_S Y$ is the fiber product of the underlying topological spaces of X, Y, and S. The induced morphism $X \times_S Y \to Y$ is again a submersion.

We will prove this proposition later in this section. Before we prove this proposition we will look at some properties a fiber product should satisfy if it exists, and study some cases in which fiber products of manifolds do not exist or behave unexpectedly.

Suppose that S is a manifold, and let $f : X \to S$ and $g : Y \to S$ be two morphisms of manifolds. Suppose, moreover, that the fiber product $X \times_S Y$ exists in the corresponding category of manifolds. As the set of points of any manifold can be identified with the set of morphisms from the one-point manifold to that manifold, one deduces that the set of points of $X \times_S Y$ is the fiber product in the category of sets:

$$|X \times_S Y| = |X| \times_{|S|} |Y| \text{ in } \mathbf{Set}.$$

Moreover, let T be the fiber product of f and g in the category **Top** of topological spaces. By the universal property of the fiber product there exists a continuous map $X \times_S Y \to T$. As the sets of points underlying T and $X \times_S Y$ both equal the fiber product in the category of sets, this map is moreover a bijection. We conclude that $X \times_S Y$ and T have the same underlying sets, and the topology on $X \times_S Y$ is stronger than the topology on T. The following example shows that this topology can be strictly stronger.

Example 1.1.8. Consider the morphism of manifolds

$$f : \mathbb{R} \to \mathbb{R} : x \mapsto \begin{cases} \exp(-1/x^2)\sin(2\pi/x) & \text{if } x \neq 0 \\ 0 & \text{if } x = 0. \end{cases}$$

Moreover, let $g : \{0\} \to \mathbb{R}$ be the inclusion. The topological fiber product of these two morphisms is then simply the subspace

$$f^{-1}(0) = \{0\} \cup \{1/n : n \in \mathbb{Z} \setminus \{0\}\} \subseteq \mathbb{R}.$$

This space, however, is not locally connected, so it cannot be (the topological space underlying) the fiber product in the category of manifolds.
In fact, one easily checks that the space $f^{-1}(0)$ equipped with the discrete topology is the fiber product of f and g in the category of manifolds. Its underlying topology is strictly stronger than the subspace topology on $f^{-1}(0) \subseteq \mathbb{R}$.

The following example shows a case in which a fiber product does not exist at all.

Example 1.1.9. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be the morphism $(x, y) \mapsto xy$, and let $g : \{0\} \to \mathbb{R}$ be the inclusion. Suppose that the fiber product F of these two morphisms exists. The underlying set is

$$\{(x,y) \in \mathbb{R}^2 : xy = 0\},$$

and the underlying topology is stronger than the subspace topology. Moreover, one can construct using the universal property a morphism $\mathbb{R} \to F : x \mapsto (x,0)$, which shows that the subset of F consisting of the x-axis is in fact homeomorphic to $\mathbb{R}$, and similarly for the y-axis; it follows that the topology on F in fact equals the subspace topology. However, this can never be the topology of a manifold. Indeed, if $U \subseteq F$ is an open neighborhood of the origin, then removing the origin from U breaks U into at least four connected components. In particular such an U can never be homeomorphic to a ball in $\mathbb{R}^n$. We must conclude that the fiber product of f and g does not exist in the category of manifolds.

Let $f : X \to S$ and $g : Y \to S$ be two morphisms of manifolds. We say that f and g are *transversal* if for all $x \in X$ and $y \in Y$ with $f(x) = g(y) =: s$ the linear map $df_x + dg_y : T_{X,x} \oplus T_{Y,y} \to T_{S,s}$ is surjective. The following lemma shows that the fiber product of f and g exists if f and g are transversal.

Lemma 1.1.10. Let $f : X \to S$ and $g : Y \to S$ be morphisms of manifolds, and assume that f and g are transversal. Then the subset

$$X \times_S Y = \{(x,y) \in X \times Y : f(x) = g(y)\} \subseteq X \times Y$$

has the structure of an embedded submanifold of dimension $\dim X + \dim Y - \dim S$, and this submanifold is the fiber product of f and g in the category of smooth manifolds. Its underlying topological space is the topological fiber product of f and g.
If $(x,y) \in X \times_S Y$ is any point, then the tangent space to $X \times_S Y$ at (x,y) is the subspace of the tangent space to $X \times Y$ at (x,y) given by

$$T_{X \times_S Y,(x,y)} = \{(v,w) \in T_{X \times Y,(x,y)} \cong T_{X,x} \times T_{Y,y} : df_x(v) = dg_y(w)\}.$$

Proof. Consider the morphism

$$h : X \times Y \to S \times S : h(x,y) = (f(x), g(y)).$$

As f and g are transversal, it is straightforward to prove that h is transversal to the inclusion map of the diagonal $\Delta \subseteq S \times S$. From [Lee03, Theorem 6.30] it follows that

$$F := h^{-1}(\Delta) = \{(x,y) \in X \times Y : f(x) = g(y)\}$$

is an embedded submanifold of $X \times Y$ whose codimension equals the dimension of S, and from [Lee03, Theorem 5.29] one deduces that F satisfies the universal property of the fiber product of f and g. Moreover, $F \subseteq X \times Y$ with its subspace topology is also the topological fiber product.

Let $(x,y) \in F$ be a point with $s := f(x) = g(y)$, and consider the linear map

$$\phi : T_{X \times Y,(x,y)} \cong T_{X,x} \times T_{Y,y} \to T_{S,s} : (v,w) \mapsto df_x(v) - dg_y(w).$$

As f and g are transversal, it follows that ϕ is surjective. The natural diagram

$$\begin{tikzcd} T_{F,(x,y)} \arrow[r, "dp_1"] \arrow[d, "dp_2"] & T_{X,x} \arrow[d, "df"] \\ T_{Y,y} \arrow[r, "dg"] & T_{S,s} \end{tikzcd} \tag{1.1.11}$$

commutes, so the tangent space $T_{F,(x,y)} \subseteq T_{X\times Y,(x,y)}$ to F at (x, y) is contained in the kernel of ϕ. By comparing dimensions we find that $T_{F,(x,y)} = \ker \phi$. ☐

The lemma allows us to prove Proposition 1.1.7.

Proof of Proposition 1.1.7. As f is a submersion, it is transversal to g. The fiber product $F = X \times_S Y$ therefore exists by Lemma 1.1.10. By chasing through diagram 1.1.11 one finds that the tangent map dp_2 is surjective, so p_2 is a submersion. ☐

Example 1.1.12. Let Y be a manifold, and let $V \subseteq Y$ be an open submanifold. Let $f : V \to Y$ denote the inclusion. Then f is a submersion. If $g : X \to Y$ is a morphism of manifolds, then the fiber product of f and g is isomorphic to the open submanifold

$$V \times_Y X = g^{-1}(V) \subseteq X.$$

Example 1.1.13. If $f : X \to Y$ is a submersion, $y \in Y$ is a point, and $g : \{y\} \to Y$ is the inclusion, then the fiber product of f and g is the fiber $X_y = f^{-1}(y)$ of f over y.

Example 1.1.14. Let X and Z be manifolds, and consider the projection $p_2 : Z \times X \to X$, which is a submersion. If $g : Y \to X$ is any other morphism, then the fiber product of p_2 and g is $Z \times Y$:

$$\begin{tikzcd} Z \times Y \arrow[r, "\mathrm{id}_Z \times g"] \arrow[d, "p_2"] \arrow[dr, phantom, "\square"] & Z \times X \arrow[d, "p_2"] \\ Y \arrow[r, "g"] & X. \end{tikzcd}$$

1.1.3 Oriented submersions

In this section we will define what it means for a submersion to have oriented fibers. Of course, we want to impose some continuity criterion on such orientations. For example, if we consider the Möbius strip as a fiber bundle over S^1 with fibers homeomorphic to $(0, 1)$, it is intuitively clear that all these fibers can be given an orientation, but these orientations can never vary continuously over the base S^1.

Recall that giving an orientation of a manifold X is equivalent to giving an orientation of its tangent bundle T_X. This leads to the following definition.

Let $f : X \to Y$ be a submersion of manifolds. The *relative tangent bundle* $T_f = T_{X/Y}$ on X is the kernel of the surjective morphism $df : T_X \to f^*T_Y$ of vector bundles on X. Note that for each $y \in Y$ the restriction of $T_{X/Y}$ to the fiber $X_y = f^{-1}(y)$ equals the tangent bundle T_{X_y}.

Definition 1.1.15. Let $f : X \to Y$ be a submersion of manifolds. An *orientation* of f (or an orientation of the fibers of f) is an orientation of the relative tangent bundle $T_{X/Y}$. An *oriented submersion* is a submersion together with an orientation.

Note that giving an orientation of the vector bundle $T_{X/Y}$ is equivalent to giving an orientation of its determinant line bundle $\det T_{X/Y} = \bigwedge^r T_{X/Y}$, where r denotes the rank of $T_{X/Y}$.

Example 1.1.16. For any manifold X giving an orientation of X is equivalent to giving an orientation of the morphism $X \to \{*\}$.

Example 1.1.17. Let F be an oriented manifold, let Y be any manifold, and define $X = F \times Y$. The projections $p_1 : X \to F$ and $p_2 : X \to Y$ induce an isomorphism $T_X \xrightarrow{\sim} p_1^*T_F \times p_2^*T_Y$, and hence an isomorphism $T_{X/Y} \xrightarrow{\sim} p_1^*T_F$. The orientation of F, therefore, induces an orientation of p_2.

Let $f : X \to Y$ and $g : Y \to S$ be two oriented submersions, with fibers of dimension r and s, respectively. The composition $gf : X \to S$ can be equipped with a canonical orientation, as follows. By using the exact sequences that define the vector bundles of f, g, and gf, we get a commutative diagram as follows:

$$\begin{tikzcd}
0 \arrow[dr] & & & 0 \arrow[d] & \\
& T_{X/S} \arrow[dr] & & f^*T_{Y/S} \arrow[d] & \\
0 \arrow[r] & T_{X/Y} \arrow[r] & T_X \arrow[r, "df"] \arrow[dr, "d(gf)"'] & f^*T_Y \arrow[r] \arrow[d, "dg"] & 0 \\
& & & f^*g^*T_S \arrow[d] \arrow[dr] & \\
& & & 0 & 0
\end{tikzcd}$$

from which we extract an exact sequence of vector bundles on X:

$$0 \to T_{X/Y} \to T_{X/S} \xrightarrow{df} f^*T_{Y/S} \to 0.$$

Choose any splitting of this exact sequence. This is always possible: we can construct a Riemannian metric on $T_{X/S}$ by using partitions of unity, and then the tangent map *df* restricts to an isomorphism of the orthogonal complement of $T_{X/Y} \subseteq T_{X/S}$ with $f^*T_{Y/S}$. Note that such a splitting is not canonical. In any case, we obtain an isomorphism

$$T_{X/Y} \oplus f^*T_{Y/S} \xrightarrow{\sim} T_{X/S},$$

and by taking determinants we get an isomorphism of line bundles

$$\det T_{X/Y} \otimes f^* \det T_{Y/S} \xrightarrow{\sim} \det T_{X/S}.$$

This latter isomorphism *is* canonical: it does not depend on the earlier choice of a splitting. The orientations of f and g, therefore, canonically define an orientation of gf.

Example 1.1.18. Let F_1 and F_2 be oriented manifolds, and let S be any manifold. Define $X = F_1 \times F_2 \times S$ and $Y = F_2 \times S$, and let $f : X \to Y$ and $g : Y \to S$ be the obvious projections. The orientations of F_1 and F_2 induce orientations of f and g (see Example 1.1.17) and hence an orientation of the projection $gf : F_1 \times F_2 \times S \to S$. This orientation agrees with the orientation of gf induced by the product orientation of $F_1 \times F_2$.

Example 1.1.19. Consider the Möbius strip $M \to S^1$ as a fiber bundle with fiber $(-1, 1)$. The submersion $M \to S^1$ does not have an orientation. Indeed, choose an orientation of S^1, and hence of the submersion $S^1 \to \{*\}$. Any orientation of the submersion $M \to S^1$ would induce an orientation of the composition $M \to S^1 \to \{*\}$, and this would yield an orientation of M. As the Möbius strip is not orientable, this cannot happen.

Example 1.1.20. Assume we have a cartesian diagram of manifolds

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

with f a submersion. Note that f' is again a submersion. The differential map $dh : T_{X'} \to h^*T_X$ restricts to an isomorphism of relative tangent bundles $T_{X'/S'} \xrightarrow{\sim} h^*T_{X/S}$. In particular, any orientation of f induces an orientation of f' in a natural way.

1.1.4 Holomorphic submersions and families of curves

A complex manifold (of dimension n) is a Hausdorff second-countable locally ringed space that is locally isomorphic to the space $\mathbb{C}^n$ with its sheaf of holomorphic functions. Equivalently, a complex manifold is a Hausdorff second-countable topological space together with an atlas of charts to opens in $\mathbb{C}^n$ whose transition functions are all holomorphic. One-dimensional complex manifolds are also called *Riemann surfaces.* As $\mathbb{C}^n \cong \mathbb{R}^{2n}$ and biholomorphic maps are diffeomorphisms, we find that every complex manifold has an underlying structure of a smooth manifold, whose (*real*) dimension is twice the dimension of the complex manifold. In this section, we will study holomorphic submersions: morphisms of complex manifolds whose underlying morphism of smooth manifolds is a submersion. It turns out that the fibers of holomorphic submersions are complex manifolds, allowing us to define families of complex manifolds.

Two morphisms $f : X \to S$ and $g : Y \to S$ of complex manifolds are *transversal* if the underlying morphisms of smooth manifolds are transversal. Analogous to Lemma 1.1.10 we can prove that if f and g are transversal, then the subset

$$X \times_S Y = \{(x, y) \in X \times Y : f(x) = g(y)\}$$

is a complex submanifold of $X \times Y$, and it is the fiber product of f and g in the category of complex manifolds. See [FG02, Chapter IV.1] for more details.

A *(holomorphic) submersion* $f : X \to Y$ of complex manifolds is a morphism of complex manifolds, such that the underlying morphism of smooth manifolds is a submersion. If $f : X \to S$ is a holomorphic submersion, then for every morphism $g : Y \to S$ of complex manifolds the fiber product $X \times_S Y$ in the category of complex manifolds exists. In particular, the fibers of a submersion are again complex manifolds: for $s \in S$, the fiber of f above s is

$$X_s = f^{-1}(s) = X \times_S \{s\}.$$

Recall that the space $\mathbb{C}^n$ is endowed with a canonical orientation. Consider the holomorphic coordinates $z_1, \ldots, z_n$ and the corresponding smooth coordinates $x_1, y_1, \ldots, x_n, y_n$ with $z_k = x_k + \sqrt{-1}y_k$. Then the canonical orientation of $\mathbb{C}^n$ is given by

$$dx_1 \wedge dy_1 \wedge \cdots \wedge dx_n \wedge dy_n.$$

Biholomorphic maps are orientation-preserving, and it follows that each complex manifold comes with a canonical orientation. The fibers of a holomorphic submersion $f : X \to Y$ are complex manifolds, so they all have a canonical orientation. By passing to coordinate charts one easily checks that these orientations define an orientation of f, called the *canonical orientation*, and this orientation is compatible with the canonical orientations of X and Y.

Definition 1.1.21. A *curve* is a compact connected Riemann surface. A *family of curves* (of genus g) is a surjective holomorphic submersion whose fibers are curves (of genus g).

It follows immediately that families of curves (of genus g) are stable under base change. This makes it possible to talk about moduli spaces of genus g curves, which we will do in Chapter 2.

Note that, by Corollary 1.1.5, the morphism of smooth manifolds underlying a family of curves is a smooth fiber bundle. However, a family of curves is not locally trivial if we consider the complex structure of its fibers. Consider the following example.

Example 1.1.22. Let $\mathbb{H} \subset \mathbb{C}$ denote the upper half space

$$\mathbb{H} = \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

The group $\mathbb{Z}^2$ acts on $\mathbb{C} \times \mathbb{H}$ by

$$(a, b) \cdot (z, \tau) = (z + a + b\tau, \tau).$$

Let $E = (\mathbb{C} \times \mathbb{H})/\mathbb{Z}^2$. This is a complex manifold, the projection $\mathbb{C} \times \mathbb{H} \to \mathbb{H}$ induces a morphism $f : E \to \mathbb{H}$, and this morphism is a family of curves of genus 1. For $\tau \in \mathbb{H}$ the fiber E_τ is the complex torus $\mathbb{C}/(\mathbb{Z}+\mathbb{Z}\tau)$. The (nonholomorphic!) diffeomorphism

$$\mathbb{C} \times \mathbb{H} \to \mathbb{C} \times \mathbb{H} : (z, \tau) \mapsto (\mathrm{Re}(z) + \mathrm{Im}(z) \cdot \tau, \tau)$$

induces an isomorphism of smooth fiber bundles

$$\begin{tikzcd}
E_i \times \mathbb{H} \arrow[rr, "\sim"] \arrow[dr, "p_2"'] & & E \arrow[dl, "f"] \\
& \mathbb{H} &
\end{tikzcd}$$

where $E_i = \mathbb{C}/(\mathbb{Z} + \mathbb{Z}i)$ denotes the fiber of $i = \sqrt{-1}$ under f.
It follows that, as a submersion of smooth manifolds, f is a trivial smooth fiber bundle, with fibers diffeomorphic to the torus. However, the fibers of f are not all mutually isomorphic as complex manifolds. The group $\mathrm{SL}_2(\mathbb{Z})$ acts on $\mathbb{H}$ by

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix} \cdot \tau = \frac{a\tau + b}{c\tau + d},$$

and the fibers over two points $\tau_1, \tau_2 \in \mathbb{H}$ are isomorphic as complex manifolds if and only if τ_1 and τ_2 lie in the same orbit of this action. See, for example, [Hai11, §1].

1.2 Currents

1.2.1 Currents on manifolds

Definition 1.2.1. Let $U \subseteq \mathbb{R}^n$ be an open subspace, and consider the set $A_c^*(U)$ of smooth forms on U with compact support. A *current* on U is then a $\mathbb{R}$-linear form

$$T : A_c^*(U) \to \mathbb{R}$$

that is continuous in the following sense: if $K \subseteq U$ is a compact subset, and $\{\omega_i\}_{i\geq 0}$ is a sequence of forms in $A_c^*(U)$ whose supports are all contained in K, such that all the coefficient functions and all their partial derivatives converge uniformly to 0, then $T(\omega_i)$ converges to zero.

In general, currents can be defined as follows.

Definition 1.2.2. Let X be a smooth manifold. A *current* on X is an $\mathbb{R}$-linear form

$$T : A_c^*(X) \to \mathbb{R}$$

such that for every open $U \subseteq X$ and every isomorphism φ of U with an open subset $U' \subseteq \mathbb{R}^n$ the composition

$$A_c^*(U') \xrightarrow{\varphi^*} A_c^*(U) \to A_c^*(X) \xrightarrow{T} \mathbb{R}$$

is a current on U'. Here the middle arrow denotes extension by zero.

The $\mathbb{R}$-vector space of currents on X is denoted by $D^*(X)$.

A current T is said to have *degree* p if $T(\omega) = 0$ for all differential q-forms with $q \neq n - p$. We denote by $D^p(X) \subseteq D^*(X)$ the subspace of degree p currents. We have a decomposition

$$D^*(X) = \bigoplus_{p \geq 0} D^p(X).$$

For each open subset $U \subseteq X$ we have an inclusion $A_c^*(U) \subseteq A_c^*(X)$ and hence a restriction map $D^*(X) \to D^*(U) : T \mapsto T|_U$. The resulting presheaf D^* of currents on X is a sheaf. For each $T \in D^*(X)$ the support $\operatorname{Supp} T$ of T is the complement of the largest open subset $U \subseteq X$ for which $T|_U = 0$. Denote by $D_c^*(X) \subseteq D^*(X)$ the subspace of currents whose support is compact.

Let $T : A_c^*(X) \to \mathbb{R}$ be a current, and let $\omega \in A^*(X)$ be a smooth form such that $\operatorname{Supp} T \cap \operatorname{Supp} \omega$ is compact. If $\{\chi_i\}_{i \in I}$ is any partition of unity of X with compact supports, then the sum

$$T(\omega) := \sum_{i \in I} T(\chi_i \omega)$$

has only finitely many nonzero terms and hence converges. It is straightforward to show that $T(\omega)$ does not depend on the chosen partition of unity. We therefore see that T extends to a linear form on the space of smooth forms ω for which $\operatorname{Supp} T \cap \operatorname{Supp} \omega$ is compact. In particular, for all $T \in D_c^*(X)$ the compactly supported current T extends to a linear form $A^*(X) \to \mathbb{R}$.

Example 1.2.3. Let X be a manifold, and let $Z \subseteq X$ be a closed oriented submanifold of codimension p. The *Dirac delta current* $\delta_Z \in D^p(X)$ associated to Z is the current defined by

$$\delta_Z(\alpha) = \int_Z \alpha|_Z \quad \text{for all } \alpha \in A_c^*(X).$$

The support of δ_Z equals Z. In particular, if Z is compact, the current δ_Z extends to a linear form on $A^*(X)$, given by the same integral formula for all $\alpha \in A^*(X)$.

Example 1.2.4. Suppose that X is an oriented manifold. If $\alpha \in A^*(X)$ is a smooth form, we define a current $[\alpha] \in D^*(X)$ as follows:

$$[\alpha](\beta) = \int_X \alpha \wedge \beta \quad \text{for all } \beta \in A_c^*(X).$$

We get an injective map $A^*(X) \to D^*(X)$. If α has degree p, then so does $[\alpha]$. The support of $[\alpha]$ equals the support of α. We therefore also obtain an injective map $A_c^*(X) \to D_c^*(X)$.

Example 1.2.5. If f is a locally integrable function on an n-dimensional oriented manifold X, then f induces a current $[f] \in D^0(X)$ given by

$$[f](\beta) = \int_X f \cdot \beta \quad \text{for all } \beta \in A_c^n(X).$$

We call a current T on an oriented manifold *smooth* if it is of the form $T = [\alpha]$ for some smooth differential form α.

Recall the exterior derivative d on the space of differential forms on X. Dually, we have a linear operator

$$b : D^*(X) \to D^*(X) : T \mapsto Td = (\omega \mapsto T(d\omega)).$$

Using Stokes' theorem, one easily proves that, given an oriented manifold X and a smooth p-form α on X, one has:

$$b[\alpha] = (-1)^{p+1}[d\alpha].$$

We define the *exterior derivative* on currents to be the linear operator d on $D^*(X)$ defined by

$$d = (-1)^{p+1}b$$

for every degree p current T. We obtain the identity

$$d[\alpha] = [d\alpha]$$

for every oriented manifold X and every smooth form α on X.

1.2.2 Pushforwards of currents

Recall that we can pull back differential forms along a morphism $f : X \to Y$ of manifolds. Dually, it is possible to push forward some currents along this morphism. Suppose that T is a current on X, and assume that the composition $\operatorname{Supp} T \hookrightarrow X \to Y$ is a proper map. If ω is any compactly supported form on Y, then

$$\operatorname{Supp}(f^*\omega) \cap \operatorname{Supp}(T) \subseteq f^{-1}(\operatorname{Supp}(\omega)) \cap \operatorname{Supp}(T)$$

is compact, and $T(f^*\omega)$ is well-defined.

Definition 1.2.6. Let $f : X \to Y$ be a morphism of manifolds, and let $T \in D^*(X)$ be a current on X. If the composition $\mathrm{Supp}(T) \to X \to Y$ is a proper map, we define the pushforward $f_*T \in D^*(Y)$ of T along f to be the current on Y given by

$$f_*T(\omega) = T(f^*\omega) \quad \text{for all } \omega \in A_c^*(Y).$$

If T is a current of degree p, then f_*T is a current of degree $p-(\dim X - \dim Y)$. We obtain a pushforward map

$$f_* : D_f^*(X) \to D^{*+\dim Y - \dim X}(Y),$$

where $D_f^*(X)$ denotes the set of currents on X whose support is proper over Y.

Example 1.2.7. Let us return to Example 1.2.3, where X is a smooth manifold and $Z \subseteq X$ is a closed oriented submanifold of codimension p. It follows that the Dirac delta current δ_Z on X can be given as the pushforward of the smooth current $[1]$ on Z:

$$\delta_Z = i_*[1]$$

where $i : Z \to X$ is the inclusion morphism.

It follows that the pushforward of a smooth current is not necessarily smooth anymore. For instance, consider the inclusion of a point $\{x\} \to X$ into a manifold with positive dimension. Then the current δ_x on X is the pushforward of the smooth form $[1]$ along the inclusion $\{x\} \to X$, but δ_x is itself not smooth: there are no smooth forms on X whose support equals $\{x\}$. If we make an additional assumption that our morphism is a submersion, then pushforwards of smooth currents along this morphism are again smooth.

Theorem 1.2.8. Let $f : X \to Y$ be an oriented submersion of oriented manifolds. If $T = [\alpha]$ is a smooth current on X whose support is proper over Y, then f_*T is a smooth current on Y.

We will postpone the proof of this theorem until Section 1.3. In this section we will find that $f_*[\alpha]$ is the current associated to the smooth form $\int_f \alpha$ on Y obtained by integrating α along the fibers of f.

Let $f : X \to Y$ be a morphism of manifolds. As d commutes with the pullback map $f^* : A^*(Y) \to A^*(X)$, it follows that the pushforward $f_* : D^*(X) \to D^*(Y)$ commutes with the operator b. Therefore, we obtain for every current $T \in D^*(X)$ whose support is proper over Y:

$$f_*dT = (-1)^{\dim(X)-\dim(Y)} df_*T.$$

1.2.3 Currents on complex manifolds

We can obtain complex-valued currents on a smooth manifold X by tensoring the space of currents $D^*(X)$ with the complex numbers:

$$D^*(X;\mathbb{C}) = D^*(X) \otimes_{\mathbb{R}} \mathbb{C}.$$

Such a current with complex coefficients can be written uniquely as $T = T_1 + \sqrt{-1} \cdot T_2$, with T_1 and T_2 currents with real coefficients. If ω is a complex-valued differential form on X with compact support, then $\omega = \omega_1 + \sqrt{-1} \cdot \omega_2$ for some real-valued differential forms ω_1, ω_2 with compact support, and

$$T(\omega) = (T_1(\omega_1) - T_2(\omega_2)) + \sqrt{-1} \cdot (T_1(\omega_2) + T_2(\omega_1)).$$

Suppose now that X is a complex manifold of (complex) dimension n. Recall that the space of complex-valued differential forms has a decomposition

$$A^*(X;\mathbb{C}) = \bigoplus_{p,q\geq 0} A^{p,q}(X).$$

Dually, the space of complex-valued currents has a decomposition

$$D^*(X;\mathbb{C}) = \bigoplus_{p,q\geq 0} D^{p,q}(X),$$

where $D^{p,q}(X)$ is dual to $A_c^{n-p,n-q}(X)$. A current $T \in D^*(X;\mathbb{C})$ is a (p,q)-*current* if it is an element of $D^{p,q}(X)$, which is the case if and only if

$$T(\omega) = 0 \quad \text{for all } \omega \in A_c^{r,s}(X;\mathbb{C}) \text{ with } (p+r, q+s) \neq (n,n).$$

Proposition 1.2.9. Let $f : X \to Y$ be a morphism of complex manifolds, and let $T \in D_f^*(X;\mathbb{C})$ be a complex-valued current on X whose support is proper over Y. If T is a (p,q)-current, then f_*T is a $(p-r, q-r)$-current, where $r = \dim(X) - \dim(Y)$.

Proof. Write $n = \dim(X)$ and $m = \dim(Y)$, so $r = n - m$. Let $\omega \in A_c^{s,t}(Y)$ with $(p-r+s, q-r+t) \neq (m,m)$. We need to prove that $(f_*T)(\omega) = 0$. We have:

$$(f_*T)(\omega) = T(f^*\omega),$$

and as T is a (p,q)-current, $f^*\omega$ is an (s,t)-form, and $(p+s, q+t) \neq (m+r, m+r) = (n,n)$, it follows that $T(f^*\omega) = 0$. $\square$

Recall, moreover, that each complex manifold X has a canonical orientation, so the notion of smooth currents exists on such a complex manifold. The inclusion $A^*(X;\mathbb{C}) \to D^*(X;\mathbb{C})$ restricts to inclusions

$$A^{p,q}(X) \to D^{p,q}(X).$$

In particular, it holds that a differential form on X is a (p,q)-form if and only if the associated current $[\alpha]$ is a (p,q)-current. This observation allows us to provide an easy proof for Proposition 1.3.19.

Finally, notice that the Dolbeault operators can be generalized to a setting of currents: for each p-current T and each smooth form α with compact support we set

$$(\partial T)(\alpha) = (-1)^{p+1} T(\partial\alpha) \quad \text{and} \quad (\overline{\partial} T)(\alpha) = (-1)^{p+1} T(\overline{\partial}\alpha).$$

Note that ∂ maps (p,q)-currents to $(p+1,q)$-currents, and $\overline{\partial}$ maps (p,q)-currents to $(p,q+1)$-currents. We have $d = \partial + \overline{\partial}$, and $\partial^2 = \overline{\partial}^2 = 0$.

1.3 Integration along fibers

In this section we will introduce the fiber integral operator along oriented submersions of manifolds. See also [Sto70, Appendix II].

If $f : X \to S$ is a submersion of manifolds, we denote by

$$A^*_f(X) \subseteq A^*(X)$$

the ideal consisting of those differential forms ω on X whose support is proper over S (that is, the composition $\mathrm{Supp}(\omega) \to X \to S$ is a proper map). Notice that the restrictions of such a form to the fibers of f are compactly supported differential forms. The following definition, therefore, makes sense.

Definition 1.3.1. Let $f : X \to S$ be an oriented submersion whose nonempty fibers have dimension r. A *fiber integral* (along f) is a linear map

$$\int_f : A^*_f(X) \to A^*(S)$$

that satisfies the following properties:

1. For any k-form $\omega \in A^k_f(X)$ with $k < r$ we have
$$\int_f \omega = 0.$$
2. For any r-form $\omega \in A^r_f(X)$ the fiber integral $\int_f \omega$ is a 0-form (so a smooth function) on S given by
$$\left(\int_f \omega\right)(s) = \int_{X_s} \omega|_{X_s} \quad \text{for all } s \in S.$$
3. $\int_f$ satisfies the *projection formula*: for all $\omega \in A^*_f(X)$ and all $\eta \in A^*(S)$ we have:
$$\int_f (\omega \wedge f^*\eta) = \left(\int_f \omega\right) \wedge \eta.$$

It turns out that, in fact, these defining properties uniquely determine a linear map $A^*_f(X) \to A^*(S)$.

Theorem 1.3.2. Let $f : X \to S$ be an oriented submersion. There exists a unique fiber integral along f.

This theorem allows us to refer to this linear map as *the* fiber integral along f. We will prove this theorem later in this section.

The fiber integral generalizes the integral operator on compactly supported smooth forms on manifolds.

Example 1.3.3. Let X be an oriented manifold. Let $f : X \to \{*\}$ be the associated oriented submersion. Note that $A^*_f(X) = A^*_c(X)$. The integral operator $\int_X : A^*_c(X) \to \mathbb{R} = A^*(\{*\})$ is a (and hence *the*) fiber integral along f.

Note that we have defined the fiber integral along $f : X \to S$ only for forms ω on X whose support is proper over the base S. A priori, it might seem sufficient for such a smooth form ω to have fiberwise compact support, which is a weaker condition than having proper support over the base. However, the forms we obtain in this way need not be smooth. This is demonstrated by the following example.

Example 1.3.4. Let $b : \mathbb{R} \to \mathbb{R}$ be a *bump function*: b is smooth, we have $b > 0$ on the interval $(-1, 1)$, and $b = 0$ outside this interval. Moreover, we normalize b in such a way that

$$\int_{\mathbb{R}} b(x)dx = 1.$$

Now consider the following smooth 1-form on $\mathbb{R} \times \mathbb{R}$:

$$\omega := yb(xy)dx.$$

Let p be the projection $\mathbb{R} \times \mathbb{R} \to \mathbb{R}$ mapping (x, y) to y. One easily checks that the restriction of ω to each fiber of p is compactly supported. However, the function

$$\left(\int_p \omega\right)(y) = \int_{x\in\mathbb{R}} yb(xy)dx = \begin{cases} -1 & \text{if } y < 0 \\ 0 & \text{if } y = 0 \\ 1 & \text{if } y > 0 \end{cases}$$

on $\mathbb{R}$ is not continuous at $y = 0$.
Indeed, the support of ω is not proper over $\mathbb{R}$: it is the closed subset

$$\operatorname{Supp} \omega = \overline{\{(x, y) : y \neq 0, |xy| < 1\}} \subset \mathbb{R} \times \mathbb{R},$$

and its fiber over the point $0 \in \mathbb{R}$ is noncompact, as this fiber equals $\mathbb{R}$.

1.3.1 First properties of the fiber integral

Before proving Theorem 1.3.2, we will deduce some properties of fiber integrals. Fix an oriented submersion $f : X \to S$, and let $\int_f : A^*_f(X) \to A^*(S)$ be a fiber integral along f.

If a form $\omega \in A^*_f(X)$ is zero on $f^{-1}(V)$ for some open $V \subseteq S$, then it is reasonable to expect the fiber integral $\int_f \omega$ to be zero on V. This is indeed the case, as the following proposition shows.

Proposition 1.3.5. For every $\omega \in A_f^*(X)$ we have:

$$\operatorname{Supp}\left(\int_f \omega\right) \subseteq f[\operatorname{Supp}\omega].$$

Proof. Note that $f[\operatorname{Supp}\omega] \subseteq S$ is closed as the composition $\operatorname{Supp}\omega \hookrightarrow X \to S$ is proper. Construct a smooth function $\chi : S \to \mathbb{R}$ with $\chi^{-1}(0) = f[\operatorname{Supp}\omega]$ (cf. [Lee03, Theorem 2.29]). As $\chi \equiv 0$ on $f[\operatorname{Supp}\omega]$, we have $f^*\chi \equiv 0$ on $\operatorname{Supp}\omega$. Now apply the projection formula:

$$\chi \cdot \int_f \omega = \int_f \omega \wedge f^*\chi = \int_f 0 = 0.$$

The support of $\int_f \omega$ must therefore be obtained in $\chi^{-1}(0) = f[\operatorname{Supp}\omega]$. ☐

The fiber integral is a linear map and therefore commutes with finite sums. A stronger statement holds.

Proposition 1.3.6. Let $\{\omega_i\}_{i\in I}$ be a family of forms in $A_f^*(X)$. Assume that the collection

$$\{f[\operatorname{Supp}\omega_i]\}_{i\in I}$$

is locally finite in S. Then the collection

$$\{\operatorname{Supp}\omega_i\}_{i\in I}$$

is locally finite in X, the sum $\sum_{i\in I}\omega_i$ has proper support over S, and we have an equality of forms on S:

$$\int_f \sum_{i\in I}\omega_i = \sum_{i\in I}\int_f \omega_i.$$

Proof. Let $K \subseteq X$ be compact. If $i \in I$ is such that $K \cap \operatorname{Supp}\omega_i$ is nonempty, then $f[K] \cap f[\operatorname{Supp}\omega_i]$ is nonempty. By assumption there can only be finitely many such $i \in I$. We conclude that $\{\operatorname{Supp}\omega_i\}_{i\in I}$ is locally finite. It follows that $\bigcup_{i\in I}\operatorname{Supp}\omega_i$ is closed in X, and it contains the support of $\sum_{i\in I}\omega_i$.

Now let $L \subseteq S$ be compact. Then there are only finitely many $i \in I$ for which $L \cap f[\operatorname{Supp}\omega_i]$ is nonempty. Therefore $f^{-1}(L) \cap \operatorname{Supp}\omega_i$ is empty for all but finitely many $i \in I$. We have:

$$f^{-1}(L) \cap \operatorname{Supp}\left(\sum_{i\in I}\omega_i\right) \subseteq \bigcup_{i\in I}(f^{-1}(L) \cap \operatorname{Supp}\omega_i).$$

The right hand side of this equation is compact: $f^{-1}(L) \cap \operatorname{Supp}\omega_i$ is compact for all $i \in I$ and nonempty for finitely many $i \in I$. We conclude that $f^{-1}(L) \cap \operatorname{Supp}\left(\sum_{i\in I}\omega_i\right)$ is compact, too, and $\sum_{i\in I}\omega_i$ has proper support over S.

Let $V \subseteq S$ be a relatively compact open subset, and let L denote its closure in S. Take a smooth function $\chi : S \to \mathbb{R}$ such that $\chi|_L \equiv 1$ and such that $\operatorname{Supp}\chi$ is compact (see [dRha84, Corollary 1]). If $i \in I$ is such that $\omega_i \wedge f^*\chi$ is nonzero, then $\operatorname{Supp}\omega_i \cap f^{-1}(\operatorname{Supp}\chi)$ is nonempty, and hence $f[\operatorname{Supp}\omega_i] \cap \operatorname{Supp}\chi$ is nonempty. By assumption, there can only be finitely many such $i \in I$. So $\omega_i \wedge f^*\chi = 0$ for all but finitely many $i \in I$. We can therefore exchange sum and integral as follows:

$$\chi \cdot \int_f \sum_{i\in I} \omega_i = \int_f \sum_{i\in I} (\omega_i \wedge f^*\chi) = \sum_{i\in I} \int_f (\omega_i \wedge f^*\chi) = \chi \cdot \sum_{i\in I} \int_f \omega_i,$$

where the first and last equalities follow from the projection formula. It follows that the restrictions of $\int_f \sum_{i\in I} \omega_i$ and $\sum_{i\in I} \int_f \omega_i$ to V are equal. As S can be covered by such relatively compact opens, the desired result follows. ☐

1.3.2 Construction of the fiber integral

In this section we will prove Theorem 1.3.2. We will first prove this theorem in the case where the base S is such that the vector bundle A^1_S is free, and then extend to the general case by gluing.

Let X be a manifold, and let $r \geq 0$ be an integer. We denote by $A^{\leq r}_X$ the subbundle of the vector bundle of differential forms A^*_X given by

$$A^{\leq r}_X = \bigoplus_{k=0}^{r} A^k_X.$$

Sections of $A^{\leq r}_X$, therefore, are finite sums of differential forms of degree at most r. Similarly, for a submersion $f : X \to S$ we define

$$A^{\leq r}_f(X) = \bigoplus_{k=0}^{r} A^k_f(X) = A^{\leq r}(X) \cap A^*_f(X) \subseteq A^*_f(X).$$

Lemma 1.3.7. Let $f : X \to S$ be a submersion whose nonempty fibers have dimension r. Assume that the vector bundle A^1_S is free: there are 1-forms $\eta_1, \ldots, \eta_n$ on S such that

$$A^1_S = A^0_S \cdot \eta_1 \oplus \cdots \oplus A^0_S \cdot \eta_n.$$

For each subset $I = \{i_1, \ldots, i_p\} \subseteq \{1, \ldots, n\}$ with $i_1 < \cdots < i_p$ define

$$\eta_I = \eta_{i_1} \wedge \cdots \wedge \eta_{i_p}.$$

Then each form $\omega \in A^*(X)$ can be written as

$$\omega = \sum_{I \subseteq \{1,\ldots,n\}} \omega_I \wedge f^*\eta_I$$

where each ω_I is an element of $A^{\leq r}(X)$.
If moreover ω has proper support over S, then we may assume each ω_I has proper support over S, too.

In order to deduce the second part of the above lemma from the first part, we will be using bump functions on X with proper support over the base S. The existence of such bump functions is guaranteed by the following lemma.

Lemma 1.3.8. Let $f : X \to S$ be a morphism of manifolds, and let $P \subseteq X$ be a closed subset such that $f|_P : P \to S$ is a proper map. Then there exists a smooth function χ on X with proper support over S such that $\chi|_P \equiv 1$.

Proof. We will construct an open subset $U \subseteq X$ such that $P \subseteq U$ and such that the closure $\bar{U}$ is proper over S. We can then let χ be any bump function for P with support in U (cf. [Lee03, Proposition 2.25]).

As S is locally compact and paracompact, there exists an open covering $S = \bigcup_{i\in I} V_i$ such that the collection $\{V_i\}_{i\in I}$ (and hence $\{\bar{V}_i\}_{i\in I}$) is locally finite and such that the closure $\bar{V}_i \subseteq S$ is compact for all $i \in I$.

For all $i \in I$ the set $f^{-1}(\bar{V}_i) \cap P$ is compact. As X is locally compact and paracompact, there exists an open $W_i \subseteq X$ that contains $f^{-1}(\bar{V}_i) \cap P$, such that the closure $\bar{W}_i$ is compact.

Now define $U_i = f^{-1}(\bar{V}_i) \cap W_i$. The collection $\{U_i\}_{i\in I}$ is easily seen to be locally finite, and the same therefore holds for the collection $\{\bar{U}_i\}_{i\in I}$. Define $U = \bigcup_{i\in I} U_i$. We have $\bar{U} = \bigcup_{i\in I} \bar{U}_i$, and we claim that $\bar{U} \to S$ is proper. If $K \subseteq S$ is compact, then

$$\begin{aligned} f^{-1}(K) \cap \bar{U} = f^{-1}(K) \cap \bigcup_{i\in I} \bar{U}_i &= \bigcup_{i\in I} (f^{-1}(K) \cap \overline{f^{-1}(\bar{V}_i) \cap W_i}) \\ &\subseteq \bigcup_{I\in I} f^{-1}(K \cap \bar{V}_i) \cap \bar{W}_i . \end{aligned}$$

As $\{\bar{V}_i\}_{i\in I}$ is locally finite, the intersection $K \cap \bar{V}_i$ is empty for all but finitely many $i \in I$. The union $\bigcup_{i\in I} f^{-1}(K \cap \bar{V}_i) \cap \bar{W}_i$, therefore, is a finite union of compact sets, and hence compact. We find that $f^{-1}(K) \cap \bar{U}$ is compact, so $\bar{U}$ is proper over S. □

Proof of Lemma 1.3.7. We have an equality of vector bundles

$$A_S^* = \bigwedge A_S^1 = \bigoplus_I A_S^0 \cdot \eta_I$$

with I ranging over all subsets of $\{1, \dots, n\}$.

Consider the subbundles $A_X^{\leq r}$ and $f^* A_S$ of the vector bundle A_X^* on X. Taking wedge products induces a morphism of vector bundles

$$A_X^{\leq r} \otimes f^* A_S^* \to A_X^* .$$

We claim that this morphism is surjective. This can be checked locally. As f locally looks like a projection $p_2 : F \times S \to S$ it suffices to prove surjectivity in the case that f is such a projection. In that case the canonical morphism

$$p_1^* A_F^* \otimes p_2^* A_S^* \to A_{F\times S}^*$$

is an isomorphism, and it factors as

$$p_1^* A_F^* \otimes p_2^* A_S^* \to A_{F\times S}^{\leq r} \otimes p_2^* A_S^* \to A_{F\times S}^*$$

since $p_1^* A_F^*$ is a subbundle of $A_{F\times S}^{\leq r}$. We see, in particular, that the canonical morphism $A_{F\times S}^{\leq r} \otimes p_2^* A_S^* \to A_{F\times S}^*$ is surjective, proving our claim.

As we are working with vector bundles over smooth manifolds, taking global sections is exact and commutes with tensor products. We therefore obtain a surjective map

$$A^{\leq r}(X) \otimes_{A^0(X)} \Gamma(f^* A_S^*) \to A^*(X)$$

induced by the wedge product. As A_S^* is free and generated by η_I, the $A^0(X)$-module $\Gamma(f^* A_S)$ is free and generated by $f^*\eta_I$. We therefore find that every element of $A^*(X)$ can be written as described in the statement of the lemma.

Suppose, moreover, that $\omega \in A_f^*(X)$ has proper support over S. Write $\omega = \sum_I \omega_I \wedge f^*\eta_I$. By Lemma 1.3.8 there is a smooth function χ on X with $\chi \equiv 1$ on $\operatorname{Supp}\omega$ such that $\operatorname{Supp}\chi$ is proper over S. We have

$$\omega = \chi\omega = \sum_I (\chi\omega_I) \wedge f^*\eta_I.$$

Note that the support of each $\chi\omega_I$ is proper over S. ☐

Lemma 1.3.9. Let $f : X \to S$ be an oriented submersion whose nonempty fibers have dimension r. Let $\omega \in A_f^{\leq r}(X)$, and consider the following function on S:

$$T(\omega) : S \to \mathbb{R} : s \mapsto \int_{X_s} \omega|_{X_s}.$$

Then $T(\omega)$ is smooth. In particular we obtain a linear map $T : A_f^{\leq r}(X) \to A^0(S)$.

Proof. As f locally looks like a projection $f : F \times S \to S$ with $F \subseteq \mathbb{R}^r$ and $S \subseteq \mathbb{R}^n$ open subsets, we may use partitions of unity to restrict to the case where f is such a projection and $\omega \in A^{\leq r}(F \times S)$ has compact support. In this specific case smoothness follows from the dominated convergence theorem. ☐

Using Lemma 1.3.7 we can now show that there exists a unique fiber integral along a submersion $X \to S$ if the base S is such that A_S^1 is free.

Lemma 1.3.10. Let $f : X \to S$ be an oriented submersion. Assume that A_S^1 is free. Then there exists a unique fiber integral operator $\int_f : A_f^*(X) \to A^*(S)$.

Proof. Let r be the dimension of the nonempty fibers of f. Let $T : A_f^{\leq r}(X) \to A^0(S)$ denote the map defined in Lemma 1.3.9. By Lemma 1.3.7 and the defining properties of the fiber integral, a fiber integral $\int_f$, if it exists, is uniquely determined by the identity

$$\int_f \omega \wedge f^*\eta = T(\omega) \cdot \eta \quad \text{for all } \omega \in A_f^{\leq r}(X), \eta \in A^*(S).$$

In order to prove existence of the fiber integral, choose 1-forms $\eta_1, \ldots, \eta_n$ on S such that

$$A^1_S = A^0_S \cdot \eta_1 \oplus \cdots \oplus A^0_S \cdot \eta_n.$$

If $\omega \in A^*_f(X)$ is any form, we can write $\omega = \sum_I \omega_I \wedge f^*\eta_I$ with $\omega_I \in A^{\leq r}_f(X)$ for all $I \subseteq \{1, \ldots, n\}$. We wish to define

$$\int_f \omega = \sum_I T(\omega_I) \cdot \eta_I \in A^*(S).$$

Of course we need to verify this does not depend on the choice of the forms ω_I. We claim that if $\omega_I \in A^{\leq r}(X)$ are such that $\sum_{i \in I} \omega_I \wedge f^*\eta_I = 0$, then the degree r part of each restriction $\omega_I|_{X_s}$ vanishes. In that case we have $T(\omega_I) = 0$ for all I, and the operator $\int_f$ is clearly well-defined.

Suppose that $\omega_I \in A^{\leq r}(X)$ are such that $\sum_I \omega_I \wedge f^*\eta_I = 0$. As the claim can be checked locally on X, and as submersions locally look like projections, we may reduce to the case where f is a projection $f : X = F \times S \to S$. Moreover, we may shrink F such that A^1_F is free. Let $\xi_1, \ldots, \xi_r \in A^1(F)$ be 1-forms such that

$$A^1_F = A^0_F \cdot \xi_1 \oplus \cdots \oplus A^0_F \cdot \xi_r.$$

We then find that $A^*(F \times S)$ is the free $A^0(F \times S)$-module generated by the forms

$$g^*\xi_J \wedge f^*\eta_K,$$

where J and K range over all subsets of $\{1, \ldots, r\}$ and $\{1, \ldots, n\}$, respectively. As each ω_I is an element of $A^{\leq r}(F \times S)$, there are (unique) smooth functions α_{IJK} on $F \times S$ such that

$$\omega_I = \sum_{\substack{J,K \\ |J|+|K| \leq r}} \alpha_{IJK} \cdot g^*\xi_J \wedge f^*\eta_K.$$

Again J and K range over the subsets of $\{1, \ldots, r\}$ and $\{1, \ldots, n\}$. We thus find:

$$0 = \sum_I \omega_I \wedge f^*\eta_I = \sum_{\substack{I,J,K \\ |J|+|K| \leq r}} \alpha_{IJK} \cdot g^*\xi_J \wedge f^*\eta_K \wedge f^*\eta_I.$$

Let $R = \{1, \ldots, r\}$, and take the $(g^*\xi_R \wedge f^*\eta_I)$-part of the above sum to obtain:

$$\alpha_{IR\emptyset} \cdot g^*\xi_R = 0,$$

so $\alpha_{IR\emptyset} = 0$. Restricting ω_I to a fiber F of f yields the form

$$\sum_J \alpha_{IJ\emptyset}|_F \cdot \xi_J.$$

Its degree r part is $\alpha_{IR\emptyset} \cdot \xi_R = 0$, which proves the claim.

We thus obtain a well-defined linear map $\int_f : A^*_f(X) \to A^*(S)$. It is straightforward to verify that it satisfies the defining properties of the fiber integral. □

The following lemma will allow us to generalize Lemma 1.3.10 to the setting of arbitrary oriented submersions by gluing.

Lemma 1.3.11. Let $f : X \to S$ be an oriented submersion, and assume that A^1_S is free. Let $V \subseteq S$ be any open subset, and define $U = f^{-1}(V)$. Let $f' : U \to V$ denote the restriction of f to U. For all $\omega \in A^*_f(X)$ we have $\omega|_U \in A^*_{f'}(U)$, and the following equality holds:

$$\left(\int_f \omega\right)\Big|_V = \int_{f'} \omega|_U.$$

Note that in this setting f' is a submersion, too, and the orientation of f induces an orientation of f'. As both A^1_S and A^1_V are free, fiber integrals along f and f' exist and are unique.

Proof. If $K \subseteq V$ is compact, then

$$(f')^{-1}(K) \cap \operatorname{Supp}(\omega|_U) = f^{-1}(K) \cap \operatorname{Supp}(\omega) \cap U = f^{-1}(K) \cap \operatorname{Supp}(\omega)$$

is compact, too, so $\omega|_U \in A^*_{f'}(U)$.

By Lemma 1.3.7 it suffices to prove the given identity for $\omega \in A^*_f(X)$ of the form $\omega = \omega' \wedge f^*\eta$, with $\omega' \in A^{\leq r}_f(X)$ and $\eta \in A^*(S)$. In this case the given identity follows directly from the defining properties of the fiber integrals. ☐

We can now prove Theorem 1.3.2.

Proof. (Proof of Theorem 1.3.2) Let $\mathcal{B}$ be the collection consisting of all opens $V \subseteq S$ for which A^1_V is free. Note that $\mathcal{B}$ is a basis for the topology of S.

We will first prove that a fiber integral $\int_f$ along f, if it exists, is necessarily unique. Let $\omega \in A^*_f(X)$. First assume that there exists some $V \in \mathcal{B}$ such that $\operatorname{Supp} \omega \subseteq f^{-1}(V)$. Write $U = f^{-1}(V)$. Lemma 1.3.7 implies that there are forms $\omega'_1, \dots, \omega'_t \in A^{\leq r}_f(U)$ and $\eta'_1, \dots, \eta'_t \in A^*(V)$ such that

$$\omega|_V = \sum_{i=1}^t \omega'_i \wedge f^*\eta'_i.$$

Let $\chi \in A^0(X)$ be a bump function for $\operatorname{Supp} \omega$ supported in U, and, likewise, let $\psi \in A^0(S)$ be a bump function for $f[\operatorname{Supp} \omega]$ supported in V. For each $1 \leq i \leq t$ let $\omega_i \in A^{\leq r}_f(X)$ be the extension by zero of $\chi|_U \cdot \omega'_i \in A^{\leq r}_f(U)$, and let $\eta_i \in A^*(S)$ be the extension by zero of $\psi|_V \cdot \eta'_i$. We then find:

$$\omega = \sum_{i=1}^t \omega_i \wedge f^*\eta_i,$$

from which we deduce that $\int_f \omega$ is uniquely determined by the defining properties of fiber integrals.

In general, let $S = \bigcup_{i \in I} V_i$ be an open covering with $V_i \in \mathcal{B}$ for all $i \in I$, and let $\{\chi_i\}_{i \in I}$ be a partition of unity subordinate to this open covering. By Proposition 1.3.6 we then have for each $\omega \in A_f^*(X)$:

$$\int_f \omega = \int_f \sum_{i \in I} f^* \chi_i \cdot \omega = \sum_{i \in I} \int_f f^* \chi_i \cdot \omega.$$

As the support of each $f^*\chi_i \cdot \omega$ is contained in $f^{-1}(V_i)$ we find that each fiber integral $\int_f f^*\chi_i \cdot \omega$ is uniquely determined, and the same therefore holds for $\int_f \omega$.

We will now construct a fiber integral along f by gluing. Consider the sheaf $f_* A_f^*$ on S given by

$$f_* A_f^*(V) = A_f^*(f^{-1}(V)).$$

We will construct a sheaf morphism $\int_f : f_* A_f^* \to A_S^*$, which in particular induces a linear map $A_f^*(X) \to A^*(S)$. Lemma 1.3.10 implies that for every $V \in \mathcal{B}$ we have a unique fiber integral operator

$$\int_{f|_{f^{-1}(V)}} : A_f^*(f^{-1}(V)) \to A^*(S).$$

Lemma 1.3.11 states that these fiber integrals are compatible with restrictions and hence define a sheaf morphism on the basis $\mathcal{B}$ and therefore a sheaf morphism $f_* A_f^* \to A_S^*$. We obtain a linear map on global sections $\int_f : A_f^*(X) \to A^*(S)$. Note that it is uniquely determined by the following property: for each open $V \in \mathcal{B}$ and each form $\omega \in A_f^*(X)$, we have

$$\left(\int_f \omega\right)\bigg|_V = \int_{f|_{f^{-1}(V)}} \omega|_{f^{-1}(V)}.$$

We can prove that $\int_f$ is in fact a fiber integral by verifying its defining properties locally. ☐

Remark 1.3.12. We have constructed the fiber integral along the oriented submersion $f : X \to S$ by 'gluing' fiber integrals along the induced submersions $f^{-1}(V) \to V$ for each open $V \subseteq S$ with A_V^1 free. Note that, in particular, this implies that the fiber integral is well-behaved with respect to restrictions to open subsets. More precisely, a stronger version of Lemma 1.3.11 holds: we no longer need to assume that A_S^1 is free.

Another approach of constructing the fiber integral is provided by Stoll [Sto70]; we will briefly sketch the construction here. Let $f : X \to S$ be a submersion, and let $r, q \geq 0$. For each point $s \in S$ Stoll then obtains a canonical linear map

$$A^{q+r}(X) \to A^r(X_s) \otimes_{\mathbb{R}} \bigwedge^q T_{S,s}^*,$$

where $X_s = f^{-1}(s)$ and where $T_{S,s}^*$ is the fiber of the cotangent bundle A_S^1 at s. If f is the projection $\mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^n$ and if we let $x_1, \ldots, x_r$ and $y_1, \ldots, y_n$ denote the standard coordinates of $\mathbb{R}^r$ and $\mathbb{R}^n$, respectively, then this linear map is given by

$$\alpha \cdot dx_I \wedge dy_J \mapsto \begin{cases} (\alpha|_{X_s} \cdot dx_I) \otimes dy_J & \text{if } |I| = r \\ 0 & \text{if } |I| < r. \end{cases}$$

In general, the above linear map restricts to a linear map

$$A_f^{q+r}(X) \to A_c^r(X_s) \otimes_{\mathbb{R}} \bigwedge^q T_{S,s}^*,$$

and composing this linear map with the integral operator $\int_{X_s} : A_c^r(X_s) \to \mathbb{R}$ finally yields a linear map

$$A_f^{q+r}(X) \to \bigwedge^q T_{S,s}^*.$$

By repeating this construction for each $s \in S$ we obtain a function that maps forms $\omega \in A_f^{r+q}(X)$ to sections of the sheaf A_S^q. Stoll then shows that these sections are smooth, and that the operator we obtain satisfies all properties one would expect of the fiber integral. It is straightforward to prove that Stoll's fiber integral matches the one defined in this section.

1.3.3 More properties of the fiber integral

In this section we will prove some more properties of the fiber integral $\int_f$ along an oriented submersion f.

Let r denote the dimension of the nonempty fibers of f. The fiber integral $\int_f$ maps k-forms with $k < r$ to zero, and it maps r-forms to smooth functions. More generally, we can show that the fiber integral maps k-forms to $(k-r)$-forms for all $k \geq 0$ (where it is understood that $A^{k-r}(S) = \{0\}$ if $k - r < 0$).

Proposition 1.3.13. Let $f : X \to S$ be an oriented submersion whose nonempty fibers have dimension r. For all $k \geq 0$ and all $\omega \in A_f^k(X)$ we have $\int_f \omega \in A^{k-r}(S)$, where $A^l(S) = 0$ for all $l < 0$.

Proof. For $k \geq 0$ we obtain a linear map $A_f^k(X) \to A^{k-r}(S)$ by composition:

$$A_f^k(X) \hookrightarrow A_f^*(X) \xrightarrow{\int_f} A^*(S) = \bigoplus_{l \in \mathbb{Z}} A^l(S) \twoheadrightarrow A^{k-r}(S).$$

Taking the direct sum over all $k \geq 0$ yields a linear map

$$A_f^*(X) \to A^{*-r}(S).$$

It is straightforward to verify that this linear map is again a fiber integral, so it must in fact be equal to $\int_f$ by Theorem 1.3.2. □

In Section 1.3.2 we have seen that the fiber integral can be computed locally on the base. This fact can be used to prove the following base change formula.

Proposition 1.3.14 (Base change formula)**.** Suppose we have a cartesian diagram of manifolds

$$\begin{tikzcd}
X' \arrow[r, "h"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

where f (and hence f') is an oriented submersion. For every form $\omega \in A^*_f(X)$, we have $h^*\omega \in A^*_{f'}(X')$, and the following identity holds:

$$\int_{f'} h^*\omega = g^* \int_f \omega.$$

Proof. A map between locally compact Hausdorff spaces is proper if and only if it is universally closed. The map $\operatorname{Supp}\omega \to S$, therefore, is universally closed, and hence the same holds for the map $h^{-1}(\operatorname{Supp}\omega) \to S'$, which is therefore proper. As $\operatorname{Supp} h^*\omega \subseteq h^{-1}(\operatorname{Supp}\omega)$ is a closed subset it follows that the induced morphism $\operatorname{Supp} h^*\omega \to S'$ is proper, so $h^*\omega \in A^*_{f'}(X')$.

As fiber integrals can be computed locally on the base, it suffices to prove the given identity in the case where A^1_S is free. In this case we may use Lemma 1.3.7 to reduce to the case where ω is of the form $\omega' \wedge f^*\eta$ with $\omega' \in A^{\leq r}_f(X)$ and $\eta \in A^*(S)$. Now the given identity follows immediately from the defining properties of the fiber integral. ☐

Lemma 1.3.15. Let $f : X \to S$ be an oriented submersion. Let $U \subseteq X$ be an open subset, and consider the induced submersion $f|_U : U \to S$. If $\omega \in A^*_f(X)$ is such that $\operatorname{Supp}\omega \subseteq U$, then

$$\int_{f|_U} \omega|_U = \int_f \omega \in A^*(S).$$

Proof. We may assume that A^1_S is free. Let r denote the dimension of the nonempty fibers of f. By Lemma 1.3.7 we can write

$$\omega = \sum_{i=1}^{t} \omega_i \wedge f^*\eta_i$$

with $\omega_i \in A^{\leq r}_f(X)$ and $\eta_i \in A^*(S)$. By using a bump function for $\operatorname{Supp}\omega$ with support in U we may assume that $\operatorname{Supp}\omega_i \subseteq U$ for all $1 \leq i \leq t$. The lemma now follows immediately from the defining properties of the fiber integral. ☐

Lemma 1.3.16. Let X and S be oriented manifolds, and let $f : X \to S$ be an oriented submersion. Assume that all orientations are compatible: we assume that the given orientation of X matches the orientation induced by the composition of the oriented submersions $X \xrightarrow{f} S \to \{*\}$. Then for all $\omega \in A_c^*(X)$ we have:

$$\int_X \omega = \int_S \int_f \omega.$$

Proof. By the base change formula and Lemma 1.3.15 we may use partitions of unity to reduce to the case where f is a projection of the form $\mathbb{R}^r \times \mathbb{R}^n \to \mathbb{R}^n$. In this specific case the above identity follows from Fubini's theorem. ☐

As a corollary we obtain Theorem 1.2.8.

Proof of Theorem 1.2.8. Let $\alpha \in A_f^*(X)$. An application of Lemma 1.3.16 and the projection formula then gives:

$$f_*[\alpha] = \left[\int_f \alpha\right].$$ ☐

Proposition 1.3.17. Let $f : X \to Y$ and $g : Y \to S$ be oriented submersions. Note that the composition $gf : X \to S$ can be endowed with a canonical orientation. For each $\omega \in A_{gf}^*(X)$ we have:

$$\omega \in A_f^*(X), \quad \int_f \omega \in A_g^*(Y), \quad \text{and} \quad \int_g \int_f \omega = \int_{gf} \omega.$$

Proof. Let $K \subseteq Y$ be compact. Then

$$\operatorname{Supp} \omega \cap f^{-1}(K) \subseteq \operatorname{Supp} \omega \cap (gf)^{-1}(f[K])$$

is a closed subset of a compact set and hence compact. We find that $\omega \in A_f^*(X)$.

Similarly, for $L \subseteq S$ compact, we see that

$$\operatorname{Supp}\left(\int_f \omega\right) \cap g^{-1}(L) \subseteq f[\operatorname{Supp} \omega] \cap g^{-1}(L) = f[\operatorname{Supp} \omega \cap (gf)^{-1}(L)]$$

is a closed subset of a compact set and therefore compact, so $\int_f \omega \in A_g^*(X)$.

In order to prove the given identity, first assume that S is orientable, and fix an orientation of S. Then the orientations of S, f, and g induce orientations of X and Y. We have an equality of smooth currents:

$$\left[\int_{gf} \omega\right] = (gf)_*[\omega] = g_*(f_*[\omega]) = g_* \left[\int_f \omega\right] = \left[\int_g \int_f \omega\right]$$

and hence an equality of the underlying differential forms.

If S is not orientable, denote by $\pi_S : \tilde{S} \to S$ its orientation double cover. Write $\tilde{X} = X \times_S \tilde{S}$ and $\tilde{Y} = Y \times_S \tilde{S}$. We obtain a commutative diagram with cartesian squares:

$$\begin{tikzcd}
\tilde{X} \arrow[r, "\tilde{f}"] \arrow[d, two heads, "\pi_X"] \arrow[dr, phantom, "\square"] & \tilde{Y} \arrow[r, "\tilde{g}"] \arrow[d, two heads, "\pi_Y"] \arrow[dr, phantom, "\square"] & \tilde{S} \arrow[d, two heads, "\pi_S"] \\
X \arrow[r, "f"] & Y \arrow[r, "g"] & S
\end{tikzcd}$$

and $\tilde{f}$, $\tilde{g}$, and $\tilde{g}\tilde{f}$ are oriented submersions. We apply the base change formula, and the fact that the proposition holds if the base is orientable:

$$\pi_S^* \int_{gf} \omega = \int_{\tilde{g}\tilde{f}} \pi_X^* \omega = \int_{\tilde{g}} \int_{\tilde{f}} \pi_X^* \omega = \int_{\tilde{g}} \pi_Y^* \int_f \omega = \pi_S^* \int_g \int_f \omega.$$

As π_S is a submersion, its associated pullback operator on differential forms is injective, and the desired result follows. ☐

Proposition 1.3.18 (Relative Stokes' Theorem)**.** Let $f : X \to S$ be an oriented submersion whose nonempty fibers have dimension r. For each $\omega \in A_f^*(X)$ we have:

$$\int_f d\omega = (-1)^r d \int_f \omega.$$

Proof. Assume that S is orientable; picking an orientation of S yields an orientation of X. Then we obtain the following equality of smooth currents for each $\omega \in A_f^p(X)$:

$$\left[\int_f d\omega\right] = f_*[d\omega] = f_* d[\omega] = (-1)^r d f_*[\omega] = (-1)^r d \left[\int_f \omega\right] = \left[(-1)^r d \int_f \omega\right].$$

The underlying smooth forms on S are, therefore, equal.

By using orientation double covers as in the proof of Proposition 1.3.17 it is possible to extend the proof to the case where S is not orientable. ☐

1.3.4 The fiber integral along holomorphic submersions

In this section we will study the fiber integral in the setting where the manifolds are complex and the submersion is a holomorphic map.

Assume that X and S are complex manifolds, and that $f : X \to S$ is a holomorphic submersion. Recall that we have decompositions

$$A^*(X; \mathbb{C}) = \bigoplus_{p,q \geq 0} A^{p,q}(X) \quad \text{and} \quad A^*(S; \mathbb{C}) = \bigoplus_{p,q \geq 0} A^{p,q}(S).$$

The fiber integral $\int_f$ respects these decompositions.

Proposition 1.3.19. Let $f : X \to S$ be a holomorphic submersion of complex manifolds whose nonempty fibers have (complex) dimension r. For each (p, q)-form

ω on X with proper support over S, the form $\int_f \omega$ is a $(p-r, q-r)$-form on S.

Proof. As ω is a (p,q)-form, the current $[\omega]$ is a (p,q)-current. By Proposition 1.2.9 the pushforward $f_*[\omega]$ is a $(p-r, q-r)$-current on S, and as $f_*[\omega] = [\int_f \omega]$ it follows that $\int_f \omega$ is a $(p-r, q-r)$-form. □

A version of the relative Stokes' Theorem 1.3.18 exists for the Dolbeault operators

$$\partial : A^{p,q}(X) \to A^{p+1,q}(X) \quad \text{and} \quad \overline{\partial} : A^{p,q}(X) \to A^{p,q+1}(X).$$

Proposition 1.3.20. Let $f : X \to S$ be a holomorphic submersion of complex manifolds. For each smooth complex-valued form ω on X with proper support over S the following identities hold:

$$\int_f \partial\omega = \partial \int_f \omega \quad \text{and} \quad \int_f \overline{\partial}\omega = \overline{\partial} \int_f \omega.$$

Proof. We may assume that ω is a form of type (p,q) for some $p, q \geq 0$. Let r denote the (complex) dimension of the nonempty fibers of f. By Proposition 1.3.18 we have

$$\int_f \partial\omega + \int_f \overline{\partial}\omega = \int_f d\omega = d\int_f \omega = \partial \int_f \omega + \overline{\partial} \int_f \omega.$$

It follows from Proposition 1.3.19 that $\int_f \partial\omega$ and $\partial \int_f \omega$ are both $(p+1-r, q-r)$-forms, whereas $\int_f \overline{\partial}\omega$ and $\overline{\partial} \int_f \omega$ are both $(p-r, q+1-r)$-forms. The desired equalities follow. □

1.4 Hermitian line bundles on families of curves

In this section we will construct several line bundles that appear canonically on families of genus g curves. Moreover we will equip these line bundles with canonical hermitian metrics. By using the Deligne pairing we will be able to exhibit some canonical isometries between these hermitian line bundles. These canonical isometries will be used in Chapter 4 to prove certain equalities in rings of tautological differential forms.

1.4.1 The Poincaré bundle

In this section we will define the Poincaré bundle on families of complex tori. We refer to [BL04], [BHdJ18] for more details.

Let T be a complex torus, and let $T^\vee = \mathrm{Pic}^0(T)$ be its dual torus. A *Poincaré bundle* on the product $T \times T^\vee$ is a line bundle $\mathcal{P}$ that satisfies the following properties:

- For each class $[L] \in T^\vee = \mathrm{Pic}^0(T)$ the pullback of $\mathcal{P}$ along the inclusion $T \to T \times T^\vee : x \mapsto (x, [L])$ is isomorphic to L;
- The pullback of $\mathcal{P}$ along the zero section $\nu_0 : T^\vee \to T \times T^\vee$ is trivial.

A Poincaré bundle always exists, and is unique up to isomorphism. A *rigidified Poincaré bundle* is a Poincaré bundle $\mathcal{P}$ together with a *rigidification*, which is an isomorphism $\nu_0^* \mathcal{P} \xrightarrow{\sim} O_{T^\vee}$. A rigidified Poincaré bundle exists and is unique up to a unique isomorphism.

Next, let $f : \mathcal{T} \to S$ be a family of complex tori. We may construct the *dual family* $f^\vee : \mathcal{T}^\vee \to S$, whose fiber $(\mathcal{T}^\vee)_s$ above a point $s \in S$ is the torus $(\mathcal{T}_s)^\vee$ dual to the torus $\mathcal{T}_s = f^{-1}(s)$. The fiber product $\mathcal{T} \times_S \mathcal{T}^\vee$ admits a *rigidified Poincaré bundle* $\mathcal{P}$, which is a line bundle $\mathcal{P}$ together with an isomorphism $\nu_0^* \mathcal{P} \xrightarrow{\sim} O_{\mathcal{T}^\vee}$ with $\nu_0 : \mathcal{T}^\vee \to \mathcal{T} \times_S \mathcal{T}^\vee$ the zero section, such that for each $s \in S$ the restriction of $\mathcal{P}$ to the fiber $(\mathcal{T} \times_S \mathcal{T}^\vee)_s = \mathcal{T}_s \times \mathcal{T}_s^\vee$ is the rigidified Poincaré bundle on that fiber. The rigidified Poincaré bundle on $\mathcal{T} \times_S \mathcal{T}^\vee$ is unique up to a unique isomorphism.

Let $f : \mathcal{C} \to S$ be a family of curves. Associated to the family f is the Jacobian family $\mathcal{J} \to S$, which is a family of complex tori whose fiber over a point $s \in S$ is the Jacobian $\mathrm{Jac}(\mathcal{C}_s)$ of the curve $\mathcal{C}_s = f^{-1}(s)$. The canonical principal polarizations of the fibers of the Jacobian family give rise to a morphism $\lambda : \mathcal{J} \to \mathcal{J}^\vee$ of families over S. We will denote by $\mathcal{P}_\lambda$ the line bundle on $\mathcal{J} \times_S \mathcal{J}$ obtained by pulling back the Poincaré bundle $\mathcal{P}$ along the morphism $\mathrm{id}_{\mathcal{J}} \times \lambda : \mathcal{J} \times_S \mathcal{J} \to \mathcal{J} \times_S \mathcal{J}^\vee$. Moreover, pulling back $\mathcal{P}_\lambda$ along the diagonal morphism $\mathcal{J} \to \mathcal{J} \times_S \mathcal{J}$ yields a line bundle on $\mathcal{J}$, the *canonical bundle* on $\mathcal{J}$, which we will denote by $\mathcal{B}$.

Recall the following: if T is a complex torus, and L is a line bundle on T, then L induces a morphism

$$\varphi_L : T \to T^\vee : x \mapsto [t_x^* L \otimes L^{\otimes -1}] \in T^\vee = \mathrm{Pic}^0(T).$$

A line bundle L on the Jacobian $J = \mathrm{Jac}(C)$ is called *polarizing* if the associated polarization $\varphi_L : J \to J^\vee$ is a multiple of the canonical polarization $\lambda : J \to J^\vee$. Equivalently, L is polarizing if and only if its first Chern class is a multiple of the Chern class $c_1(O(\Theta))$ of any theta divisor Θ on J.

Proposition 1.4.1. Let C be a curve. The canonical bundle $\mathcal{B}$ on the Jacobian $\mathrm{Jac}(C)$ is polarizing.

We will use the following lemma.

Lemma 1.4.2. Let T be a complex torus, and let $\lambda : T \to T^\vee$ be any homomorphism of complex tori. Let L be the pullback of the Poincaré bundle $\mathcal{P}$ along the morphism $(\mathrm{id}_T, \lambda) : T \to T \times T^\vee$. Then we have an equality of homomorphisms

$$\varphi_L = \lambda + \lambda^\vee \circ \kappa : T \to T^\vee,$$

where $\kappa : T \to T^{\vee\vee}$ is the canonical isomorphism.

Proof. It is a routine exercise to verify that the following diagram is commutative.

$$\begin{tikzcd}
T \arrow[r, "\varphi_L"] \arrow[d, "(\mathrm{id},\lambda)"'] & T^\vee & T^\vee \times T \arrow[l, "\mathrm{id}_{T^\vee} + \lambda^\vee \kappa"'] \arrow[d, "\mathrm{id}_{T^\vee} \times \kappa"] \\
T \times T^\vee \arrow[r, "\varphi_{\mathcal{P}}"] & (T \times T^\vee)^\vee \arrow[u, "(\mathrm{id},\lambda)^\vee"] \arrow[r, "\sim"] & T^\vee \times T^{\vee\vee} \arrow[ul, "\mathrm{id}_{T^\vee} + \lambda^\vee"']
\end{tikzcd}$$

Moreover, the composition

$$T \times T^\vee \xrightarrow{\varphi_{\mathcal{P}}} (T \times T^\vee)^\vee \xrightarrow{\sim} T^\vee \times T^{\vee\vee} \xrightarrow{\mathrm{id}_{T^\vee} \times \kappa^{-1}} T^\vee \times T$$

is the isomorphism $T \times T^\vee \to T^\vee \times T$ that swaps coordinates; see, for instance, [BL04, Exercise 2.16]. The desired result now follows. ☐

Proof of Proposition 1.4.1. Let Θ be any theta divisor on $\mathrm{Jac}(C)$, and let $\lambda = \varphi_{O(\Theta)}$ be the canonical principal polarization of $\mathrm{Jac}(C)$. By Lemma 1.4.2 and [BL04, Corollary 2.4.6(c)] we have:

$$\varphi_{\mathcal{B}} = \lambda + \lambda^\vee \circ \kappa = 2\lambda,$$

so $c_1(\mathcal{B}) = 2c_1(O(\Theta))$. ☐

1.4.2 The Deligne pairing

This section serves to introduce the Deligne pairing, which is a pairing associated to a family $f : \mathcal{C} \to S$ of curves that maps a pair of line bundles on $\mathcal{C}$ to a line bundle on S. The Deligne pairing will be used to construct isomorphisms between line bundles that appear canonically on families of curves. We refer to [Del87], [ACG11] for a more detailed treatment.

Let C be a curve. Suppose that f is a nonzero meromorphic function on C, and let $D = \sum_{x \in C} n_x \cdot x$ be a divisor of C such that $\mathrm{div}\, f$ and D are disjoint. We then define

$$f[D] := \prod_{x \in C} f(x)^{n_x} \in \mathbb{C} \setminus \{0\}.$$

To any two line bundles L, M on C we assign a vector space $\langle L, M \rangle$ as follows. Denote by V the complex vector space whose basis consists of pairs (l, m), where l and m are nonzero meromorphic sections of L and M respectively whose divisors are disjoint. The vector space $\langle L, M \rangle$, then, is the quotient of V modulo the subspace spanned by the relations

$$(fl, m) - f[\mathrm{div}\, m] \cdot (l, m) \quad \text{and} \quad (l, gm) - g[\mathrm{div}\, l] \cdot (l, m),$$

where f and g are meromorphic functions on C, such that $\mathrm{div}\, f$ and $\mathrm{div}\, m$ are disjoint, and $\mathrm{div}\, g$ and $\mathrm{div}\, l$ are disjoint. The image of a pair $(l, m) \in V$ under the quotient map $V \to \langle L, M \rangle$ is denoted by $\langle l, m \rangle$. Using Weil reciprocity one can show that $\langle L, M\rangle$ is a one-dimensional vector space. We call the vector space $\langle L, M\rangle$ the *Deligne pairing* of L and M.

We can generalize the Deligne pairing to families as follows. Assume that $f : \mathcal{C} \to S$ is a family of curves, and let L, M be holomorphic line bundles on $\mathcal{C}$. Then L and M induce a holomorphic line bundle $\langle L, M\rangle_f$ on S. The fiber of $\langle L, M\rangle_f$ at a point $s \in S$ is the Deligne pairing $\langle L_s, M_s\rangle$ of the restrictions of L and M to the curve $\mathcal{C}_s$. If $U \subseteq S$ is an open subset, and l and m are nonzero meromorphic sections of $L|_{f^{-1}(U)}$ and $M|_{f^{-1}(U)}$ whose divisors are disjoint and do not contain any of the fibers of f, then

$$\langle l, m\rangle : s \mapsto \langle l(s), m(s)\rangle$$

is a generating section of $\langle L, M\rangle_f|_U$. We will often omit the subscript and write $\langle L, M\rangle$ instead of $\langle L, M\rangle_f$ if the morphism f is clear from the context.

For holomorphic line bundles L, L_1, L_2, M on $\mathcal{C}$ we have canonical isomorphisms

$$\begin{aligned}
\langle L, M\rangle &\xrightarrow{\sim} \langle M, L\rangle : & \langle l, m\rangle &\mapsto \langle m, l\rangle \\
\langle L_1, M\rangle \otimes \langle L_2, M\rangle &\xrightarrow{\sim} \langle L_1 \otimes L_2, M\rangle : & \langle l_1, m\rangle \otimes \langle l_2, m\rangle &\mapsto \langle l_1 \otimes l_2, m\rangle \\
\langle O_{\mathcal{C}}, M\rangle &\xrightarrow{\sim} O_S : & \langle 1, m\rangle &\mapsto 1 \\
\langle L^{\otimes -1}, M\rangle &\xrightarrow{\sim} \langle L, M\rangle^{\otimes -1} : & \langle l^{\otimes -1}, m\rangle &\mapsto \langle l, m\rangle^{\otimes -1}
\end{aligned}$$

where for every nonzero vector v in a one-dimensional vector space V the vector $v^{\otimes -1} \in V^{\otimes -1} = V^\vee$ denotes the vector dual to v.

Isomorphisms $L_1 \xrightarrow{\sim} L_2$ and $M_1 \xrightarrow{\sim} M_2$ induce isomorphisms $\langle L_1, M\rangle \xrightarrow{\sim} \langle L_2, M\rangle$ and $\langle L, M_1\rangle \xrightarrow{\sim} \langle L, M_2\rangle$, respectively. If $\sigma : S \to \mathcal{C}$ is a section of f, we denote by $O(\sigma)$ the line bundle on $\mathcal{C}$ associated to the divisor $\sigma[S] \subseteq \mathcal{C}$. We have a canonical isomorphism

$$\langle O(\sigma), L\rangle \xrightarrow{\sim} \sigma^* L : \langle 1, l\rangle \mapsto \sigma^* l.$$

Moreover, suppose that the degree of the restriction of L to each fiber of f equals d. Then for each line bundle N on S we have a canonical isomorphism

$$\langle L, f^* N\rangle \xrightarrow{\sim} N^{\otimes d} : \langle l, f^* n\rangle \mapsto n^{\otimes d}.$$

Finally, the Deligne pairing is well-behaved with respect to base change. More precisely: if we have a cartesian diagram

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "h"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, "f"] \\
S' \arrow[r, "g"] & S
\end{tikzcd}$$

where f (and hence f') is a family of curves, and L and M are line bundles on $\mathcal{C}$, then there is a canonical isomorphism

$$g^* \langle L, M\rangle_f \xrightarrow{\sim} \langle h^* L, h^* M\rangle_{f'}.$$

The Deligne pairing and the Poincaré bundle are related as follows. We say that a line bundle L on $\mathcal{C}$ has *relative degree 0* (with respect to $f : \mathcal{C} \to S$) if its restriction to each fiber of f has degree 0. If L is a line bundle on $\mathcal{C}$ of relative degree 0, then L induces a section $[L] : S \to \mathcal{J}$ of the Jacobian family $\mathcal{J} \to S$. The morphism $[L]$ maps a point $s \in S$ to the class of the restriction $L|_{\mathcal{C}_s}$ in $\mathcal{J}_s = \mathrm{Jac}(\mathcal{C}_s)$.

Proposition 1.4.3 ([Mor85])**.** Let L and M be line bundles on $\mathcal{C}$ of relative degree 0. Then there is a canonical isomorphism

$$\langle L, M \rangle \xrightarrow{\sim} ([L],[M])^* \mathcal{P}_\lambda^{\otimes -1}.$$

In particular, for any line bundle L on $\mathcal{C}$ with relative degree 0 one has a canonical isomorphism

$$\langle L, L \rangle \simeq ([L],[L])^* \mathcal{P}_\lambda^{\otimes -1} = [L]^* \mathcal{B}^{\otimes -1}. \qquad \square$$

Let $\omega = \omega_{\mathcal{C}/S}$ denote the relative holomorphic cotangent bundle of f. This is a line bundle on $\mathcal{C}$ whose restriction to each fiber $\mathcal{C}_s$ is the sheaf $\Omega^1_{\mathcal{C}_s}$ of holomorphic 1-forms on $\mathcal{C}_s$. It can be obtained as follows. Let $T^{1,0}_{\mathcal{C}}$ and $T^{1,0}_S$ denote the holomorphic tangent bundles of $\mathcal{C}$ and S. The tangent map $df : T^{1,0}_{\mathcal{C}} \to f^* T^{1,0}_S$ is surjective since f is a submersion, and its kernel is a line bundle $T^{1,0}_{\mathcal{C}/S}$, the *relative holomorphic tangent bundle* of f. Its dual is the line bundle $\omega_{\mathcal{C}/S}$. In other words: as f is a submersion, we can view the pullback $f^*\Omega^1_S$ along f of the bundle Ω^1_S of holomorphic 1-forms as a subbundle of $\Omega^1_{\mathcal{C}}$, and $\omega_{\mathcal{C}/S}$ is the quotient bundle $\Omega^1_X / f^*\Omega^1_S$. There is a canonical isomorphism of line bundles on $\mathcal{C}$:

$$\Delta^* O(\Delta) \xrightarrow{\sim} \omega^{\otimes -1},$$

where $\Delta : \mathcal{C} \to \mathcal{C} \times_S \mathcal{C}$ is the diagonal morphism.

Consider the fiber product $\mathcal{C}^2 = \mathcal{C} \times_S \mathcal{C}$ and let $p_1, p_2 : \mathcal{C}^2 \to \mathcal{C}$ be the two projections. Notice that p_1 and p_2 are families of genus g curves. Consider the following line bundle on $\mathcal{C}^2$:

$$O((2g-2)\Delta) \otimes p_2^* \omega^{\otimes -1}.$$

This line bundle has relative degree 0 with respect to p_1 and hence induces a morphism

$$\kappa := \mathcal{C} \to \mathcal{J} : x \mapsto [O((2g-2)x) \otimes \omega^{\otimes -1}] \in \mathrm{Jac}(\mathcal{C}_{f(x)}),$$

and by Proposition 1.4.3 there is a canonical isomorphism of line bundles on $\mathcal{C}$:

$$\left\langle O((2g-2)\Delta) \otimes p_2^*\omega^{\otimes -1}, O((2g-2)\Delta) \otimes p_2^*\omega^{\otimes -1} \right\rangle_{p_1} \xrightarrow{\sim} \kappa^* \mathcal{B}^{\otimes -1}.$$

By the bilinearity of the Deligne pairing, the left hand side of this isomorphism is canonically isomorphic to

$$\langle O(\Delta), O(\Delta) \rangle_{p_1}^{\otimes (2g-2)^2} \otimes \langle O(\Delta), p_2^*\omega \rangle_{p_1}^{\otimes -2(2g-2)} \otimes \langle p_2^*\omega, p_2^*\omega \rangle_{p_1}.$$

Recall that there are canonical isomorphisms

$$\langle O(\Delta), O(\Delta)\rangle_{p_1} \xrightarrow{\sim} \Delta^* O(\Delta) \xrightarrow{\sim} \omega^{\otimes -1} \quad \text{and} \quad \langle O(\Delta), p_2^*\omega\rangle_{p_1} \xrightarrow{\sim} \Delta^* p_2^*\omega = \omega,$$

and as the Deligne pairing is well-behaved with respect to base change

$$\langle p_2^*\omega, p_2^*\omega\rangle_{p_1} \xrightarrow{\sim} f^*\langle \omega, \omega\rangle_f.$$

By piecing all canonical isomorphisms together we obtain the following result.

Proposition 1.4.4. Let $f : \mathcal{C} \to S$ be a family of curves, and let $\mathcal{J} \to S$ be the corresponding Jacobian family. Let $\kappa : \mathcal{C} \to \mathcal{J}$ be the morphism that maps a point $x \in \mathcal{C}_s$ to the class $[O((2g-2)x) \otimes \omega^{\otimes -1}] \in \mathrm{Jac}(\mathcal{C}_s)$. Then we have a canonical isomorphism

$$\kappa^* \mathcal{B}^{\otimes -1} \xrightarrow{\sim} \omega^{\otimes -2g(2g-2)} \otimes f^*\langle \omega, \omega\rangle_f. \qquad \square$$

Likewise, we may consider the two sections $\sigma_1, \sigma_2 : \mathcal{C}^2 \to \mathcal{C}^3$ of the projection $p_{12} : \mathcal{C}^3 \to \mathcal{C}^2$ given by

$$\sigma_i : \mathcal{C}^2 \to \mathcal{C}^3 : (x_1, x_2) \mapsto (x_1, x_2, x_i).$$

These sections induce a line bundle $O(\sigma_2 - \sigma_1) = O(\sigma_2) \otimes O(\sigma_1)^{\otimes -1}$ on $\mathcal{C}^3$ with relative degree 0 with respect to p_{12}. Analogous to Proposition 1.4.4 we obtain the following identity.

Proposition 1.4.5. Let $f : \mathcal{C} \to S$ be a family of curves, and let $\mathcal{J} \to S$ be the corresponding Jacobian family. Let $\delta : \mathcal{C}^2 \to \mathcal{J}$ denote the morphism that maps a pair $(x, y) \in \mathcal{C}_s^2$ to the class $[O(y - x)] \in \mathrm{Jac}(\mathcal{C}_s)$. Then we have a canonical isomorphism

$$\delta^* \mathcal{B}^{\otimes -1} \xrightarrow{\sim} p_1^*\omega^{\otimes -1} \otimes p_2^*\omega^{\otimes -1} \otimes O(\Delta)^{\otimes -2}. \qquad \square$$

By applying Proposition 1.4.5 we moreover obtain a canonical isomorphism

$$\left\langle \delta^* \mathcal{B}^{\otimes -1}, \delta^* \mathcal{B}^{\otimes -1} \right\rangle_{p_1} \xrightarrow{\sim} \omega^{\otimes 4g} \otimes f^*\langle \omega, \omega\rangle_f. \tag{1.4.6}$$

Combining this canonical isomorphism with the one from 1.4.4 finally yields yet another canonical isomorphism

$$\left\langle \delta^* \mathcal{B}^{\otimes -1}, \delta^* \mathcal{B}^{\otimes -1} \right\rangle_{p_1} \otimes \kappa^* \mathcal{B} \xrightarrow{\sim} \omega^{\otimes 4g^2}. \tag{1.4.7}$$

1.4.3 Hermitian metrics, first Chern form

Let X be a complex manifold, and let E be a holomorphic vector bundle on X. A *hermitian metric* on E consists of a hermitian inner product $\langle \cdot, \cdot \rangle_x$ on each fiber E_x of E, such that these inner products vary smoothly with $x \in X$: if $U \subseteq X$ is open and $\sigma, \tau \in E(U)$ are sections, then the function

$$\langle \sigma, \tau \rangle : U \to \mathbb{C} : x \mapsto \langle \sigma(x), \tau(x) \rangle_x$$

is smooth. A *hermitian vector bundle* is a vector bundle equipped with a hermitian metric.

If σ is a holomorphic section of a hermitian vector bundle E, its norm is the real-valued function $\|\sigma\|$ on X given by

$$\langle\sigma\rangle(x) = \langle\sigma(x), \sigma(x)\rangle_x^{1/2}.$$

Conversely, if L is a holomorphic *line* bundle on X, with a generating section σ, and $f : X \to \mathbb{R}_{>0}$ is a positive-valued smooth function, then there exists a unique hermitian metric on L for which $\|\sigma\| = f$.

If V and W are two complex vector spaces with hermitian inner products $\langle\cdot,\cdot\rangle_V$ and $\langle\cdot,\cdot\rangle_W$, the tensor product $V \otimes W$ has a canonical hermitian inner product $\langle\cdot,\cdot\rangle_{V\otimes W}$ given by

$$\langle v_1 \otimes w_1, v_2 \otimes w_2\rangle_{V\otimes W} = \langle v_1, v_2\rangle_V \cdot \langle w_1, w_2\rangle_W.$$

More generally, the tensor product of two hermitian vector bundles is canonically a hermitian vector bundle.

Likewise, for any hermitian line bundle L on X, the dual line bundle $L^{\otimes -1}$ has a unique hermitian metric for which the canonical isomorphism $L \otimes L^{\otimes -1} \xrightarrow{\sim} O_X$ is an isometry, where O_X is endowed with the canonical metric $\|1\| = 1$.

Let L be a line bundle on the complex manifold X with a hermitian metric $\|\cdot\|$. The *first Chern form* $c_1(L)$ is the differential $(1,1)$-form on X defined locally by

$$c_1(L, \|\cdot\|) = \frac{\partial\overline{\partial}}{2\pi i} \log \|\sigma\|^2$$

where σ is any local generating section of L. As $\partial\overline{\partial} \log |f| = 0$ for every holomorphic function f, it follows that the first Chern form is well-defined. It is a closed real differential form whose De Rham cohomology class matches the first Chern class of the line bundle L under the isomorphism $H^2_{\mathrm{dR}}(X) \xrightarrow{\sim} H^2(X;\mathbb{R})$; see [GH94, Chapter 1.1]. If the metric $\|\cdot\|$ is clear from the context we will often omit it from our notation and write $c_1(L)$ instead of $c_1(L, \|\cdot\|)$.

The operator mapping a hermitian line bundle to its first Chern form is linear: if L_1, L_2 are two hermitian line bundles on X, then

$$c_1(L_1 \otimes L_2) = c_1(L_1) + c_1(L_2) \in A^2(X).$$

Theorem 1.4.8 (Poincaré–Lelong formula, [GH94, p. 388])**.** Let L be a hermitian line bundle on a complex manifold X and let s be a meromorphic section of L. Then we have an equality of 2-currents on X:

$$\frac{\partial\overline{\partial}}{\pi\sqrt{-1}}[\log \|s\|] + \delta_{\mathrm{div}\, s} = c_1(L, \|\cdot\|). \qquad \square$$

Note that the Poincaré–Lelong formula is stated in [GH94] in a less general setting, where L is assumed to be the trivial bundle with its canonical metric $\|1\| = 1$, and s is assumed to be a holomorphic function on X. By working locally one easily deduces Theorem 1.4.8 from the version in [GH94].

Lemma 1.4.9. Let C be a curve and let L be a hermitian line bundle on C. Then

$$\int_C c_1(L) = \deg L.$$

Proof. Take any nonzero meromorphic section s of L. By using the Poincaré–Lelong formula and evaluating the resulting 2-currents in the form $1 \in A^0_c(C)$ we obtain the desired result. ☐

The Poincaré–Lelong formula implies the following lemma, which will be used in Chapter 4.

Lemma 1.4.10. Let $f : \mathcal{C} \to S$ be a family of curves, and let L be a hermitian line bundle on $\mathcal{C}$. Then the fiber integral of the 2-form $c_1(L)$ along f is the function $S \to \mathbb{R}$ given by

$$\left(\int_f c_1(L)\right)(s) = \deg L|_{\mathcal{C}_s} \quad \text{for all } s \in S.$$

In particular, the induced function $\int_f c_1(L) : S \to \mathbb{Z} : s \mapsto \deg L|_{\mathcal{C}_s}$ is locally constant.

Proof. As $c_1(L)$ is a smooth real 2-form and f is a proper submersion with fibers of real dimension 2, the fiber integral $\int_f c_1(L)$ is a smooth function on S given by

$$\left(\int_f c_1(L)\right)(s) = \int_{\mathcal{C}_s} c_1(L)|_{\mathcal{C}_s} = \int_{\mathcal{C}_s} c_1(L|_{\mathcal{C}_s}) = \deg L|_{\mathcal{C}_s},$$

where the final equality follows from Lemma 1.4.9. This fiber integral is a smooth integer-valued function, so it is in particular locally constant. ☐

1.4.4 Admissible metrics

Line bundles on curves and complex tori admit many hermitian forms. In this section we restrict our study to *admissible* hermitian metrics. We will follow [Mor85].

Let T be a complex torus, and let L be a line bundle on T. A hermitian metric on L is *admissible* if its first Chern form is translation-invariant. The admissible metrics on O_T are exactly the constant metrics. Every line bundle L admits admissible metrics, and any two such metrics are equal up to a multiplicative constant. If L_1 and L_2 are endowed with admissible metric, then the induced metric on $L_1 \otimes L_2$ is admissible, too. If $f : T' \to T$ is a morphism of complex tori, and L is a line bundle on T with an admissible metric, then the induced metric on f^*L is admissible. See [Mor85, §3] for more details.

Example 1.4.11. Let T be a complex torus, and let $\mathcal{P}$ be the rigidified Poincaré bundle on the product torus $T \times T^\vee$. Then any two admissible metrics on $\mathcal{P}$ are equal up to a multiplicative constant. Fixing any admissible metric on $\mathcal{P}$ induces an admissible, and hence constant, metric on $O_{T^\vee}$ via the fixed isomorphism $\nu_0^*\mathcal{P} \xrightarrow{\sim} O_{T^\vee}$. It follows that there exists a unique admissible metric on $\mathcal{P}$ for which the fixed isomorphism $\nu_0^*\mathcal{P} \xrightarrow{\sim} O_{T^\vee}$ is an isometry if we endow $O_{T^\vee}$ with the canonical metric $\|1\| = 1$. We call this metric the *canonical metric* on $\mathcal{P}$. More generally, if $f : \mathcal{T} \to S$ is a family of complex tori, then there is a unique hermitian metric on the rigidified Poincaré bundle $\mathcal{P}$ on $\mathcal{T} \times_S \mathcal{T}^\vee$ whose restriction to each fiber $\mathcal{T}_s \times \mathcal{T}_s^\vee$ above S is the canonical metric on the rigidified Poincaré bundle on $\mathcal{T}_s \times \mathcal{T}_s^\vee$. See [BHdJ18, Proposition 2.8]. This metric, too, is called the *canonical metric* on $\mathcal{P}$.
If $f : \mathcal{C} \to S$ is a family of curves with Jacobian family $\mathcal{J} \to S$, then the canonical metric on the Poincaré bundle $\mathcal{P}$ on $\mathcal{J} \times_S \mathcal{J}^\vee$ and the canonical principal polarization $\lambda : \mathcal{J} \to \mathcal{J}^\vee$ induce canonical metrics on the line bundles $\mathcal{P}_\lambda = (\mathrm{id}, \lambda)^*\mathcal{P}$ and $\mathcal{B} = \Delta^*\mathcal{P}_\lambda$, and these metrics are fiberwise admissible.

Now let C be a curve of genus $g > 0$. The g-dimensional complex vector space $\Omega^1(X)$ of holomorphic 1-forms on C is endowed with an inner product

$$\langle \omega, \eta \rangle = \frac{\sqrt{-1}}{2} \int_C \omega \wedge \bar{\eta} \quad \text{for all } \omega, \eta \in \Omega^1(C).$$

Fix any orthonormal basis $\omega_1, \ldots, \omega_g$ of $\Omega^1(C)$. The *canonical 2-form* of C is the form

$$\mu := \frac{\sqrt{-1}}{2g} \sum_{i=1}^{g} \omega_i \wedge \bar{\omega}_i.$$

The canonical 2-form is a real form that satisfies

$$\int_C \mu = 1.$$

It does not depend on the choice of an orthonormal basis, and by the Riemann-Roch theorem it is a volume form.

The canonical 2-form can also be obtained from the canonical metric on $\mathcal{B}$ as follows: for $x \in C$ any point we have

$$2g\mu = c_1(j_x^*\mathcal{B}) = j_x^*(c_1(\mathcal{B})),$$

with $j_x : C \to J : y \mapsto [O(y - x)]$ the Abel–Jacobi map. Note that the 2-form $j_x^*(c_1(\mathcal{B}))$ does not depend on the choice of $x \in C$, since $c_1(\mathcal{B})$ is translation-invariant.

Let L be a line bundle on C. A hermitian metric $\|\cdot\|$ on L is *admissible* if its first Chern form $c_1(L, \|\cdot\|)$ is a multiple of the canonical 2-form μ. Note that $\|\cdot\|$ is admissible if and only if $c_1(L, \|\cdot\|) = \deg(L) \cdot \mu$, since $\int_C c_1(L, \|\cdot\|) = \deg L$. The admissible metrics on the trivial bundle O_C are precisely the constant metrics. If

L_1, L_2 are endowed with admissible metrics, then the induced metric on $L_1 \otimes L_2$ is again admissible.

If $x \in C$ is a point, and M is a *polarizing* line bundle on the Jacobian $J = \mathrm{Jac}(C)$ with an admissible metric, then the induced metric on the pullback $j_x^* M$ along the Abel–Jacobi morphism $j_x : C \to J$ is again admissible.

Remark 1.4.12. If the genus of C equals 1, then we may view C both as a curve and a complex torus via the Abel–Jacobi map, and we have two definitions for admissible metrics on line bundles on C. As the canonical 2-form on C is translation-invariant, these definitions agree.

1.4.5 Biadmissible metrics

Let C be a curve of genus g, and let L be a line bundle on the product $C \times C$. A hermitian metric on L is *biadmissible* if its restrictions to the fibers of the projection maps $p_1, p_2 : C \times C \to C$ are all admissible. Any two biadmissible metrics on L are equal up to a positive multiplicative constant, and biadmissible metrics are well-behaved with respect to tensor products. If M is a line bundle on C with an admissible metric, then the induced metrics on $p_1^* M$ and $p_2^* M$ are biadmissible.

Recall from Proposition 1.4.5 that we have a canonical isomorphism of line bundles on $C \times C$:

$$\delta^* \mathcal{B} \xrightarrow{\sim} p_1^* \omega \otimes p_2^* \omega \otimes O(\Delta)^{\otimes 2}.$$

As the canonical bundle $\mathcal{B}$ is polarizing, it is straightforward to verify that the canonical metric on $\delta^* \mathcal{B}$ is biadmissible. If ω is endowed with an admissible metric, then the induced metrics on $p_1^* \omega, p_2^* \omega$ are biadmissible, and the above isomorphism induces a biadmissible metric on $O(\Delta)^{\otimes 2}$ and therefore on $O(\Delta)$. Notice, moreover, that this metric on $O(\Delta)$ is symmetric. We thus obtain a canonical map

$$\{\text{admissible metrics on } \omega\} \to \{\text{biadmissible metrics on } O(\Delta)\}.$$

As this map is compatible with the free and transitive action of the multiplicative group $\mathbb{R}_{>0}$ on both sets, it is a bijection. In order to find the inverse of this bijection, notice that restricting the above isomorphism to the diagonal yields the canonical isomorphism

$$O_C \simeq \Delta^* \delta^* \mathcal{B} \xrightarrow{\sim} \omega^{\otimes 2} \otimes \Delta^* O(\Delta)^{\otimes 2},$$

and the induced metric on $O_C \simeq \Delta^* \delta^* \mathcal{B}$ is the canonical metric $\|1\| = 1$. Every biadmissible metric on $O(\Delta)$, therefore, induces an admissible metric on ω via the canonical isomorphism $\Delta^* O(\Delta) \xrightarrow{\sim} \omega^{\otimes -1}$.

Let $\|\cdot\|$ be any biadmissible metric on $O(\Delta)$. Taking the norm of the canonical global section 1 of $O(\Delta)$ yields a function $G = \|1\| : C \times C \to \mathbb{R}_{\geq 0}$. The function G has the following properties:

1. G is smooth and positive-valued outside the diagonal, and vanishes on the diagonal. If z is a local coordinate on an open $U \subseteq C$, then on $U \times U$ the function G can be expressed as
$$G(x, y) = |z(x) - z(y)| \cdot u(x, y)$$
with u a smooth and positive-valued function on $U \times U$.
2. G is symmetric: $G(x, y) = G(y, x)$ for all $x, y \in C$.
3. For each point $x \in X$ we have an equality of 2-currents on C:
$$\frac{\partial\overline{\partial}}{\pi\sqrt{-1}}[\log G(x, \cdot)] = \mu - \delta_x,$$
by the Poincaré–Lelong formula.

Conversely, every function $G : C \times C \to \mathbb{R}_{\geq 0}$ that satisfies these properties determines a biadmissible metric on $O(\Delta)$. According to Arakelov [Ara74] the function

$$C \to \mathbb{R} : x \mapsto \int_{y \in C} \log G(x, y)\mu(y)$$

is constant. The *Arakelov–Green function* G is the unique function that satisfies the above properties and the normalizing condition

$$\int_{y \in C} \log G(x, y)\mu(y) = 0 \quad \text{for all } x \in X.$$

We will call the biadmissible metric on $O(\Delta)$ it determines the *canonical metric* on $O(\Delta)$. The canonical metric on $O(\Delta)$ determines an admissible metric on ω via the canonical isomorphism $\Delta^* O(\Delta) \xrightarrow{\sim} \omega^{\otimes -1}$, which we will also call the *canonical metric* on ω.

Finally, if $x \in C$ is any point, restricting the canonical metric on $O(\Delta)$ to the fiber $C \times \{x\}$ yields an admissible metric on the line bundle $O(x)$, given by

$$\|1\|(y) = G(x, y).$$

We call this metric the *canonical metric* on $O(x)$. More generally, for $D = \sum_x n_x x$ a divisor on C, we obtain a *canonical metric* on the line bundle $O(D)$ via the canonical isomorphism

$$\bigotimes_{x \in X} O(x)^{\otimes n_x} \xrightarrow{\sim} O(D).$$

1.4.6 Canonical isometries

Let C be a curve with Jacobian J. We have defined canonical metrics on the line bundles $\mathcal{B}$ on J, $O(\Delta)$ on $C \times C$, and ω and $O(D)$ (with D a divisor) on C. In this section we will show that the canonical isomorphisms we obtained in Section 1.4.2 are in fact isometries. The canonical isomorphism

$$\delta^* \mathcal{B} \xrightarrow{\sim} p_1^* \omega \otimes p_2^* \omega \otimes O(\Delta)^{\otimes 2}$$

is an isometry by construction of the metrics on ω and $O(\Delta)$. For the other isomorphisms we will be using the Deligne pairing.

Let L, M be hermitian line bundles on C. To the hermitian metrics on L, M we associate a norm on the vector space $\langle L, M\rangle$:

$$\begin{aligned}\log \|\langle l, m\rangle\| &:= (\log \|m\|)[\operatorname{div}(l)] + [\log \|l\|](c_1(M)) \\ &= (\log \|l\|)[\operatorname{div}(m)] + [\log \|m\|](c_1(L)),\end{aligned}$$

where the second equality can be proved using Stokes' theorem.

More generally, let $f : \mathcal{C} \to S$ be a family of curves, and let L, M be hermitian line bundles on $\mathcal{C}$. The induced metrics on the fibers of the Deligne pairing $\langle L, M\rangle$ induce a hermitian metric on $\langle L, M\rangle$. See also [Del87, §6].

For hermitian line bundles L, L_1, L_2, M on $\mathcal{C}$ the canonical isomorphisms

$$\begin{aligned}\langle L, M\rangle &\xrightarrow{\sim} \langle M, L\rangle \\ \langle L_1, M\rangle \otimes \langle L_2, M\rangle &\xrightarrow{\sim} \langle L_1 \otimes L_2, M\rangle \\ \langle O_{\mathcal{C}}, M\rangle &\xrightarrow{\sim} O_S \\ \left\langle L^{\otimes -1}, M\right\rangle &\xrightarrow{\sim} \langle L, M\rangle^{\otimes -1}\end{aligned}$$

are isometries, where $O_{\mathcal{C}}$ and O_S are endowed with the canonical metrics given by $\|1\| = 1$.

Likewise, if N is a hermitian line bundle on S, and L a hermitian line bundle on $\mathcal{C}$ whose restriction to each fiber of f has degree d, then the canonical isomorphism

$$\langle L, f^* N\rangle \xrightarrow{\sim} N^{\otimes d}$$

is an isometry.

If we have a cartesian diagram

$$\begin{tikzcd} \mathcal{C}' \arrow[r, "h"] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, "f"] \\ S' \arrow[r, "g"] & S \end{tikzcd}$$

where f and f' are families of curves, and L and M are hermitian line bundles on $\mathcal{C}$, then the canonical isomorphism

$$g^* \langle L, M\rangle_f \xrightarrow{\sim} \langle h^* L, h^* M\rangle_{f'}$$

is an isometry.

Proposition 1.4.13 ([Del87, §6])**.** For any two hermitian line bundles L, M on $\mathcal{C}$ we have

$$c_1(\langle L, M\rangle) = \int_f c_1(L) \wedge c_1(M). \qquad \square$$

Proposition 1.4.14 ([HdJ15, Corollary 4.2])**.** Let L and M be line bundles on $\mathcal{C}$ of relative degree 0. Then the canonical isomorphism

$$\langle L, M\rangle \xrightarrow{\sim} ([L],[M])^{*}\mathcal{P}_{\lambda}^{\otimes -1}$$

from Proposition 1.4.3 is an isometry. □

Finally, assume that C is a curve, let $x \in C$ be a point, and endow $O(x)$ with its canonical metric. Then for each *admissible* line bundle M on C the canonical isomorphism

$$\langle O(x), M\rangle \xrightarrow{\sim} M_x$$

is easily seen to be an isometry. Likewise, if we equip the diagonal bundle $O(\Delta)$ on $C \times C$ with its canonical metric, and if M is a line bundle on C whose restriction to each fiber of p_1 is admissible, then the canonical isomorphism

$$\langle O(\Delta), M\rangle \xrightarrow{\sim} \Delta^{*}M$$

is an isometry. Later in this section we will generalize this statement to include arbitrary sections of families of curves.

From the above canonical isometries involving the Deligne pairing and the computations of the canonical isomorphisms 1.4.4, 1.4.6, and 1.4.7, we deduce the following result.

Proposition 1.4.15. Let C be a curve with Jacobian J, and endow the line bundles $O(\Delta)$ on $C \times C$, ω on C, and $\mathcal{B}$ on J with their canonical metrics. Then the canonical isomorphisms

$$\begin{aligned}
\kappa^{*}\mathcal{B}^{\otimes -1} &\xrightarrow{\sim} \omega^{\otimes -2g(2g-2)} \otimes f^{*}\langle \omega, \omega\rangle_{f} \\
\delta^{*}\mathcal{B}^{\otimes -1} &\xrightarrow{\sim} p_1^{*}\omega^{\otimes -1} \otimes p_2^{*}\omega^{\otimes -1} \otimes O(\Delta)^{\otimes -2} \\
\left\langle \delta^{*}\mathcal{B}^{\otimes -1}, \delta^{*}\mathcal{B}^{\otimes -1}\right\rangle_{p_1} &\xrightarrow{\sim} \omega^{4g} \otimes f^{*}\langle \omega, \omega\rangle_{f} \\
\left\langle \delta^{*}\mathcal{B}^{\otimes -1}, \delta^{*}\mathcal{B}^{\otimes -1}\right\rangle_{p_1} \otimes \kappa^{*}\mathcal{B} &\xrightarrow{\sim} \omega^{\otimes 4g^{2}}
\end{aligned}$$

are isometries. □

Notice that, in particular, the canonical metric on ω can be obtained from the canonical metric on $\mathcal{B}$ by taking pullbacks along δ and κ and Deligne pairings along p_1.

Corollary 1.4.16. Let $f : \mathcal{C} \to S$ be a family of curves, and consider the relative dualizing sheaf $\omega = \omega_{\mathcal{C}/S}$ on $\mathcal{C}$ and the diagonal bundle $O(\Delta)$ on $\mathcal{C} \times_S \mathcal{C}$.
There exists a unique fiberwise admissible hermitian metric on ω whose restriction to each fiber $\mathcal{C}_s$ of f is the canonical metric on $\omega|_{\mathcal{C}_s} = \omega_C$.
Likewise, there exists a unique fiberwise biadmissible hermitian metric on $O(\Delta)$ whose restriction to each fiber $(\mathcal{C} \times_S \mathcal{C})_s = \mathcal{C}_s \times \mathcal{C}_s$ of the morphism $\mathcal{C} \times_S \mathcal{C} \to S$

is the canonical metric on the diagonal bundle on $\mathcal{C}_s \times \mathcal{C}_s$.

As the reader may already expect, we will call these metrics on ω and $O(\Delta)$ the *canonical metrics.*

Proof. Let $\mathcal{J} \to S$ be the Jacobian family of f, and let $\mathcal{B}$ be the Poincaré bundle on $\mathcal{J}$ with its canonical metric. The canonical isomorphism

$$\left\langle \delta^*\mathcal{B}^{\otimes -1}, \delta^*\mathcal{B}^{\otimes -1} \right\rangle_{p_1} \otimes \kappa^*\mathcal{B} \xrightarrow{\sim} \omega^{\otimes 4g^2}$$

then defines a hermitian metric on $\omega^{\otimes 4g^2}$, and hence on ω. By Proposition 1.4.15 the restriction of this hermitian metric to each fiber $\mathcal{C}_s$ of f is equal to the canonical metric on $\omega_{\mathcal{C}_s}$.

Likewise, the hermitian metrics on $\mathcal{B}$ and ω determine a hermitian metric on $O(\Delta)$ via the canonical isomorphism

$$\delta^*\mathcal{B}^{\otimes -1} \xrightarrow{\sim} p_1^*\omega^{\otimes -1} \otimes p_2^*\omega^{\otimes -1} \otimes O(\Delta)^{\otimes -2},$$

and Proposition 1.4.15 ensures that the restriction of this metric to each fiber is indeed the canonical metric. ☐

Let $f : \mathcal{C} \to S$ be a family of curves, and let $\sigma : S \to \mathcal{C}$ be a section. We will endow the line bundle $O(\sigma) = O(\sigma[S])$ on $\mathcal{C}$ with a canonical metric, as follows. Both squares in the following diagram are cartesian:

$$\begin{tikzcd}
S \arrow[r, "\sigma"] \arrow[d, "\sigma"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, "\Delta"] \\
\mathcal{C} \arrow[r, "{(\sigma f, \mathrm{id}_{\mathcal{C}})}"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & \mathcal{C} \times_S \mathcal{C} \arrow[d, "p_1"] \\
S \arrow[r, "\sigma"] & \mathcal{C}
\end{tikzcd}$$

We therefore have a canonical isomorphism

$$O(\sigma) \xrightarrow{\sim} (\sigma f, \mathrm{id}_{\mathcal{C}})^* O(\Delta),$$

and the canonical metric on $O(\Delta)$ canonically induces a metric on $O(\sigma)$, which we will call the *canonical metric.* Notice that for each point $s \in S$ restricting the line bundle $O(\sigma)$ on $\mathcal{C}$ to the fiber $\mathcal{C}_s$ yields the line bundle $O(\sigma(s))$ with its canonical metric. Moreover, if M is a line bundle on $\mathcal{C}$ with a hermitian metric that is fiberwise (with respect to f) admissible, then the canonical isomorphism

$$\langle O(\sigma), M \rangle \xrightarrow{\sim} \sigma^* M$$

is an isometry.

Chapter 2

The moduli space of genus g curves

Moduli spaces can be thought of as spaces that parametrize objects of a certain type. For example, if V is a complex vector space and $k \geq 0$ an integer, the Grassmannian $G_{k,V}$ is a complex manifold whose underlying set is the set of k-dimensional linear subspaces of V, so $G_{k,V}$ parametrizes k-linear subspaces of V. There is a universal vector bundle $E_{k,V} \to G_{k,V}$ that induces every other family of k-dimensional subspaces via base change. By studying this universal family we can make statements that are valid 'universally' among families of k-dimensional subspaces of V. In Section 2.1 we will see which cohomology classes occur universally among such families.

This thesis is aimed at the moduli space $\mathcal{M}_g$ of compact Riemann surfaces of genus $g \geq 2$ and the universal family $\mathcal{C}_g \to \mathcal{M}_g$. Unfortunately, such a fine moduli space does not exist in the category of complex manifolds. In Section 2.2 we will see why the existence of nontrivial automorphisms on genus g curves prevents the existence of a fine moduli space for genus g curves.

Riemann surfaces were first studied by Riemann [Rie51; Rie57] in the context of multi-valued functions on the complex plane. Riemann already knew heuristically that a compact Riemann surface of genus $g \geq 2$ depends on $3g - 3$ parameters, or in a more modern terminology, that the moduli space $\mathcal{M}_g$ should be $(3g - 3)$-dimensional. Teichmüller [Tei44] made this statement more formal. He realized that it is impossible to endow the moduli space of genus g curves $\mathcal{M}_g$ with a well-behaved complex structure, as this space has certain singularities. He therefore constructed a covering $\mathcal{T}_g$ of $\mathcal{M}_g$ whose points are isomorphism classes of genus g curves with Teichmüller structure, and endowed this space with the structure of a complex manifold. Moreover, Teichmüller constructed a family $\mathcal{X}_g \to \mathcal{T}_g$ that is universal in the sense that any other family of genus g curves with Teichmüller structure can be obtained from this universal family by base change. He remarked that the complex manifold $\mathcal{T}_g$ is $(3g - 3)$-dimensional, and hence gave a formal meaning to Riemann's heuristic argument.

In a series of 10 talks at Henri Cartan's seminar, Grothendieck [Gro60] reformulated Teichmüller's results in a language of algebraic geometry. More precisely, he proved that the functor from complex analytic spaces to sets mapping a complex analytic space S to the set of isomorphism classes of families of genus g curves with Teichmüller structure over S is representable by the Teichmüller space $\mathcal{T}_g$. This means that $\mathcal{T}_g$ is the fine moduli space for families of genus g curves with Teichmüller structure. In Section 2.3 we will discuss the Teichmüller space $\mathcal{T}_g$.

Another approach at tackling the moduli space $\mathcal{M}_g$ was made by Deligne and Mumford [DM69]. They view $\mathcal{M}_g$ as a *stack*, rather than a complex manifold. This is the approach we will also be taking in this thesis. This approach will give us a moduli space $\mathcal{M}_g$ and a universal family $\mathcal{C}_g \to \mathcal{M}_g$. Although these are not honest complex manifolds, some theory about complex manifolds generalizes to stacks. For example, it is still possible to define differential forms and hermitian line bundles on stacks, which we will do in Sections 2.5 and 2.6.

As it turns out, it is possible to understand these differential forms and hermitian vector bundles without understanding much about the underlying stacks at all. Any reader who is not comfortable with (or interested in) using stacks can read Section 2.1 for a motivation, and afterwards read Proposition 2.5.10 and Example 2.6.1 to get some intuition for working with differential forms and vector bundles on moduli stacks.

In Chapter 4 we will often treat $\mathcal{M}_g$ and $\mathcal{C}_g$ as if they were honest complex manifolds. In such cases, the reader should understand that there is an argument being made 'behind the scenes': the given statements hold universally for families of genus g curves, and hence on the moduli stacks themselves.

In this section we will be working over the category **CMan** of complex manifolds. We also fix a Grothendieck topology on **CMan**, where a collection of morphisms $\{X_i \to X\}$ is a covering if and only if all these morphisms are open immersions and their images cover X. It now makes sense to talk about stacks over **CMan**. In this entire section every stack is assumed to be over **CMan**, so we will often abbreviate 'stack over **CMan**' to 'stack'.

2.1 Motivating example: the Grassmannian manifold

To motivate our study of families and moduli spaces of curves, we first look at a simpler and better understood example of a moduli space, the Grassmannian. We will see that studying moduli spaces can yield information on properties that hold universally on the families they classify. We refer to [BT82, §23] and [GH94, §1.5] for a more detailed treatment of the material in this section.

Definition 2.1.1. Let V be a complex vector space. A *family of k-dimensional subspaces of V* over a complex manifold S is a holomorphic sub-vector bundle $f : E \to S$ of the trivial vector bundle $V \times S \to S$, such that every fiber of f is k-dimensional.

For instance, consider the complex manifold $S = \mathbb{P}^n_{\mathbb{C}}$ whose points correspond to lines through the origin in $\mathbb{C}^{n+1}$. Let $E \to S$ be the subbundle of $\mathbb{C}^{n+1} \times S \to S$

whose fiber over a point $s \in S$ is the line in $\mathbb{C}^{n+1}$ that corresponds to s. Then $E \to S$ is a family of one-dimensional subspaces of $\mathbb{C}^{n+1}$.

More generally, for all $k \geq 0$ and all complex vector spaces V, we can consider the *Grassmannian* manifold $G_{k,V}$. Its underlying set is the set of k-dimensional subspaces of V:

$$|G_{k,V}| = \{W \subseteq V : \dim(W) = k\}$$

For example, the Grassmannian $G_{1,\mathbb{C}^{n+1}}$ is the projective space $\mathbb{P}^n_{\mathbb{C}}$. Recall that the complex structure on $\mathbb{P}^n_{\mathbb{C}}$ is constructed by gluing affine charts; the complex structure on general Grassmannians is constructed in a similar way.

The Grassmannian $G_{k,V}$ comes with a canonical family of k-dimensional subspaces of V. It is the subbundle $u : E_{k,V} \to G_{k,V}$ of the trivial bundle $V \times G_{k,V} \to G_{k,V}$ whose fiber over a point in $G_{k,V}$ equals the corresponding k-dimensional subspace of V.

Now let us assume that $f : E \to S$ is any family of k-dimensional subspaces. Then associated to f we have a morphism $\Phi_f : S \to G_{k,V}$, which maps any point $s \in S$ to the fiber $E_s \in G_{k,V}$. Moreover, the bundle $E \to S$ is the pullback of the canonical bundle $E_{k,V} \to G_{k,V}$ along the morphism Φ_f:

$$\begin{tikzcd}
E \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & E_{k,V} \arrow[d, "u"] \\
S \arrow[r, "\Phi_f"] & G_{k,V}.
\end{tikzcd}$$

It follows that the family $u : E_{k,V} \to G_{k,V}$ induces every other family $f : E \to S$ by pullback along a *unique* morphism $\Phi_f : S \to G_{k,V}$. We therefore call $u : E_{k,V} \to G_{k,V}$ the *universal* family of k-dimensional subspaces of V. We say that $G_{k,V}$ is a *fine moduli space* for k-dimensional subspaces of V.

Suppose that $f : E \to S$ is any family of k-dimensional subspaces of V. Associated to f we have some cohomology classes on S, the Chern classes

$$c_1(E), \ldots, c_k(E) \in H^*(S).$$

Moreover, we have a vector bundle Q over S defined by the following exact sequence

$$0 \to E \to V \times S \to Q \to 0.$$

Associated to Q we have some more cohomology classes on S:

$$c_1(Q), \ldots, c_{n-k}(Q) \in H^*(S),$$

where $n = \dim(V)$. These classes have the following relation:

$$(1 + c_1(E) + \cdots + c_k(E))(1 + c_1(Q) + \cdots + c_{n-k}(Q)) = 1.$$

Moreover, these cohomology classes behave well with respect to base change: if $g : T \to S$ is any morphism, and $g^*E \to T$ is the pullback of $E \to S$ along g, then $g^*Q \to T$ is the quotient bundle associated to $g^*E \to T$, and we have equalities

$$c_i(g^*E) = g^*c_i(E) \quad \text{and} \quad c_i(g^*Q) = g^*c_i(Q).$$

In particular we have for each family $f : E \to S$ of k-dimensional subspaces of V equalities of Chern classes

$$c_1(E) = \Phi_f^* c_1(E_{k,V}) \quad \text{and} \quad c_1(Q) = \Phi_f^* c_1(Q_{k,V}),$$

where $Q_{k,V}$ is the universal quotient bundle on $G_{k,V}$ defined by the exact sequence

$$0 \to E_{k,V} \to V \times G_{k,V} \to Q_{k,V} \to 0.$$

It follows that these Chern classes are in some sense *universal* on families of subspaces, and the relation we found among them is a *universal* relation. One might wonder if there are any more such universal classes or relations on families of subspaces. We can answer this question by studying the cohomology ring of the Grassmannian. Indeed, any cohomology class on the Grassmannian yields a cohomology class on the base of every family of subspaces $f : E \to S$ via pullback along Φ_f. Conversely, every universal class on bases of families of subspaces gives in particular a class on the base of the bundle $u : E_{k,V} \to G_{k,V}$. The cohomology of the Grassmannian is

$$H^*(G_{k,V}) = \frac{\mathbb{Z}[c_1(E), \ldots, c_k(E), c_1(Q), \ldots, c_{n-k}(Q)]}{((1 + c_1(E) + \cdots + c_k(E))(1 + c_1(Q) + \cdots + c_{n-k}(Q)) - 1)},$$

where $E = E_{k,V}$ and $Q = Q_{k,V}$ is the associated quotient bundle. In particular, it follows that there are no further cohomology classes or relations that are universal on families of subspaces.

Similarly, we can study other types of objects, such as Chow classes or differential forms, that are universal on families of subspaces simply by studying these objects on the Grassmannian.

The main takeaway from this section is the following.

> *Making statements about (objects on) moduli spaces is equivalent to making statements that hold universally among the families these moduli spaces classify.*

2.2 Fine moduli spaces

In Section 2.1 we constructed the Grassmannian $G_{k,V}$ that parametrizes k-dimensional subspaces of a complex vector space V, together with a universal family $u : E_{k,V} \to G_{k,V}$ that induces every other family of k-dimensional subspaces of V via base change. We called $G_{k,V}$ a *fine moduli space* for families of k-dimensional subspaces of V. In this section we will generalize this discussion. We will start with some abstract nonsense, and then apply this to some concrete examples, such as the Grassmannian we studied in Section 2.1.

Definition 2.2.1. Let **C** be a category, let **Set** denote the category of sets, and let $F : \mathbf{C} \to \mathbf{Set}$ be a contravariant functor. A *representation* of F consists of an object $\mathcal{M}$ of **C** together with a natural isomorphism τ from F to the functor of points $\mathcal{M}(-) = \mathrm{Hom}_{\mathbf{C}}(-, \mathcal{M})$. In this case, we say that $\mathcal{M}$ is a *fine moduli space* for F.

It follows from Yoneda's lemma that a representation is unique up to a unique isomorphism. In particular a fine moduli space is unique up to isomorphism.

Assume that F is representable, and fix a representation $\tau : F \xrightarrow{\sim} \mathcal{M}(-)$. Then under the bijection $\tau_{\mathcal{M}} : F(\mathcal{M}) \xrightarrow{\sim} \mathcal{M}(\mathcal{M})$, the identity $\mathrm{id}_{\mathcal{M}} : \mathcal{M} \to \mathcal{M}$ corresponds to an element $u \in F(\mathcal{M})$. We call this element the *universal element.*

Let $\alpha : S \to \mathcal{M}$ be a morphism. We obtain a commutative diagram of sets:

$$\begin{tikzcd}
F(\mathcal{M}) \arrow[r, "\tau_{\mathcal{M}}", "\sim"'] \arrow[d, "\alpha^*"] & \mathcal{M}(\mathcal{M}) \arrow[d, "- \circ \alpha"] \\
F(S) \arrow[r, "\tau_S", "\sim"'] & \mathcal{M}(S),
\end{tikzcd}$$

where we denote by α^* the map $F(\alpha) : F(\mathcal{M}) \to F(S)$ induced by $\alpha : S \to \mathcal{M}$. By chasing through this diagram we find that $\tau_S(\alpha^* u) = \alpha$. In particular, for every $f \in F(S)$ there exists a *unique* morphism $\Phi_f : S \to \mathcal{M}$ (namely $\Phi_f = \tau_S(f)$) for which $\Phi_f^* u = f$. In other words: for every object S of $\mathbf{C}$ and every element $f \in F(S)$ we can obtain f from the universal element u by pulling u back along a unique morphism $\Phi_f : S \to \mathcal{M}$.

For example, let k be a nonnegative integer, and let V be a complex vector space. Consider the following contravariant functor from the category of complex manifolds to the category of sets:

$$\begin{aligned} F : \mathbf{CMan} &\to \mathbf{Set} \\ S &\mapsto \{\text{families } E \to S \text{ of } k\text{-dimensional subspaces of } V\}. \end{aligned}$$

A morphism of complex manifolds $T \to S$ is mapped to the pullback operator that transforms families over S into families over T. We claim that the Grassmannian $G_{k,V}$ is a fine moduli space for F. Indeed, for any complex manifold S we define a map

$$\tau_S : F(S) \to G_{k,V}(S) = \mathrm{Hom}(S, G_{k,V})$$

that sends a family $f : E \to S$ to the morphism $\Phi_f : S \to G_{k,V}$ given by $\Phi_f(s) = f^{-1}(s) \subseteq V$ for all $s \in S$. Notice that τ_S is in fact a bijection. The maps τ_S induce a natural isomorphism $\tau : F \to G_{k,V}(-)$.

The universal element of the functor F is the universal family $u : E_{k,V} \to G_{k,V}$. Indeed, under the bijection $\tau_{G_{k,V}} : F(G_{k,V}) \to G_{k,V}(G_{k,V})$ this family is mapped to the identity $G_{k,V} \to G_{k,V}$. It follows once again that every family $E \to S$ of k-dimensional subspaces of V can be obtained from the universal family by taking its pullback along a unique morphism $\Phi_f : S \to G_{k,V}$.

Analogous to the Grassmannian we would like to construct a moduli space $\mathcal{M}_g$ that classifies genus g curves for a fixed integer $g \geq 0$. As there are too many genus g curves to fit into a set, we cannot expect the points of $\mathcal{M}_g$ to correspond bijectively with genus g curves. Our next best bet is to try to construct a moduli space $\mathcal{M}_g$ whose points correspond to *isomorphism classes* of genus g curves. We proceed as follows.

Two families $f : X \to S$ and $f' : X' \to S$ are *isomorphic* if there exists an isomorphism $g : X \to X'$ with $f' \circ g = f$. We can then consider the following functor:

$$\begin{aligned} F : \mathbf{CMan} &\to \mathbf{Set} \\ S &\mapsto \{\text{families } X \to S \text{ of genus } g \text{ curves}\}/\cong . \end{aligned}$$

Let us assume that F is representable by a complex manifold $\mathcal{M}_g$. Under the bijection $F(\mathcal{M}_g) \xrightarrow{\sim} \mathcal{M}_g(\mathcal{M}_g)$ the identity $\mathcal{M}_g \to \mathcal{M}_g$ corresponds to a universal family $p : \mathcal{C}_g \to \mathcal{M}_g$ of genus g curves. The bijection $F(*) \xrightarrow{\sim} \mathcal{M}_g(*)$ gives us a bijective correspondence between the points of $\mathcal{M}_g$ and isomorphism classes of genus g curves. If $f : X \to S$ is a family of genus g curves, then the associated morphism $\Phi_f : S \to \mathcal{M}_g$ maps a point $s \in S$ to the point of $\mathcal{M}_g$ that corresponds to the isomorphism class of the curve X_s, and we obtain a cartesian diagram

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & \mathcal{C}_g \arrow[d, "p"] \\ S \arrow[r, "\Phi_f"'] & \mathcal{M}_g . \end{tikzcd}$$

Now let $f : X \to S$ be an *isotrivial* family of genus g curves. That is, the fibers of f are pairwise isomorphic. Then the induced morphism $\Phi_f : S \to \mathcal{M}_g$ is constant, and factors over the singleton manifold. We obtain the following diagram with cartesian squares:

$$\begin{tikzcd} X \arrow[r] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & C \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_g \arrow[d, "p"] \\ S \arrow[r] \arrow[rr, bend right, "\Phi_f"'] & \{*\} \arrow[r] & \mathcal{M}_g \end{tikzcd}$$

where C is a genus g curve that is isomorphic to the fibers of f. We therefore see that f is a trivial family: it is isomorphic to the family $C \times S \to S$.

So the existence of a moduli space of genus g curves would imply that every isotrivial family of genus g curves is trivial. However, as the next proposition states, it is possible to construct nontrivial isotrivial families, and thus show that a fine moduli space $\mathcal{M}_g$ cannot exist.

Proposition 2.2.2. Let $g \geq 0$ be an integer. There exists an isotrivial family of genus g curves which is not trivial. In particular, there is no fine moduli space $\mathcal{M}_g$ of genus g curves in the category of complex manifolds.

Let us first prove this proposition in the case $g = 0$.

Proof for $g = 0$. First, assume that $g = 0$. Consider the projective plane $\mathbb{P}^2$ and fix a point $x \in \mathbb{P}^2$, and blow up the plane at this point. In other words, the set S

of lines through x can be identified with $\mathbb{P}^1$ and hence be given the structure of a complex manifold, and we consider the complex manifold

$$X = \{(y, \ell) \in \mathbb{P}^2 \times S : y \in \ell\}.$$

The morphism $f : X \to S : (y, \ell) \mapsto \ell$ is an isotrivial family of genus 0 curves: we have $f^{-1}(\ell) \cong \ell \cong \mathbb{P}^1$ for all $\ell \in S$. However, we claim that f is not a trivial family. If it were, it would have to be isomorphic to $\mathbb{P}^1 \times S \cong \mathbb{P}^1 \times \mathbb{P}^1$. One can show, for instance by using intersection theory, that this is not the case. The exceptional locus

$$E = \{(x, \ell) : \ell \in S\} \subseteq X$$

is a prime divisor of X with self-intersection -1 ([Har77, Proposition V.3.1]), whereas $\mathbb{P}^1 \times \mathbb{P}^1$ can be shown not to have any such prime divisors ([Har77, Example V.1.4.3]). □

For genus $g > 0$ we can construct nontrivial isotrivial families by taking a genus g curve C with a nontrivial automorphism and using this automorphism to 'twist' C. Compare this to the construction of the Möbius strip by twisting a line segment onto itself. We will finish the proof of this proposition in the next section.

2.3 Mapping class groups and Teichmüller structures

In this section we will prove Proposition 2.2.2, proving that there is no fine moduli space for genus g curves. The problem here is that genus g curves admit 'too many' automorphisms, allowing us to twist trivial families into nontrivial isotrivial families. Teichmüller [Tei44] realized this and added extra structures (which we now call *Teichmüller structures*) to the curves we are trying to classify. He thus obtained a universal family $\mathcal{X}_g \to \mathcal{T}_g$ of genus g curves with Teichmüller structures. Grothendieck [Gro60] was able to rephrase Teichmüller's results in a language of algebraic geometry. We will first discuss the results from Teichmüller and Grothendieck, and finish the section by proving that a fine moduli space $\mathcal{M}_g$ does not exist in the category of complex manifolds.

For a more detailed treatment of the material in this section we refer to Grothendieck [Gro60]; see also [AJP16] for a survey of Grothendieck's work on Teichmüller theory.

If X', X are two topological spaces, we denote by $I(X', X)$ the set of homeomorphisms $X' \to X$ modulo homotopy. If $X' = X$ then composition induces a group structure on $I(X, X)$; the resulting group is called the *mapping class group* of X and denoted $\mathrm{MCG}(X)$. In general $\mathrm{MCG}(X)$ acts from the *left*[1] on $I(X', X)$ by composition.

---

[1] In fact, Grothendieck considers the *right* action of $\mathrm{MCG}(X)$ on $I(X, X')$. In the proof of Lemma 2.3.1 we will be using the left action of the mapping class group to construct a monodromy representation. As taking inverses yields a bijective correspondence between left and right actions we lose no information if we consider left actions instead.

Assume, now, that $f : X \to S$ is a fiber bundle whose fiber F is a finite simplicial complex. The disjoint union of sets

$$\bigsqcup_{s \in S} I(X_s, F)$$

has a natural structure of a $\mathrm{MCG}(F)$-covering space over S. We denote this topological space by $\mathcal{R}(X/S)$. For fixed F and S the assignment $X \mapsto \mathcal{R}(X/S)$ is functorial. Moreover it is compatible with base changes of fiber bundles: for a continuous map $T \to S$ we have

$$\mathcal{R}(X \times_S T/T) = \mathcal{R}(X/S) \times_S T.$$

Similarly, if F is a compact connected oriented manifold, then we can consider the group $\mathrm{MCG}^+(F)$ of F consisting of homotopy classes of *orientation-preserving* homeomorphisms $F \to F$, which is a subgroup of $\mathrm{MCG}(F)$ of index at most 2. In this context the group $\mathrm{MCG}(F)$ is often called the *extended mapping class group* of F, and $\mathrm{MCG}^+(F)$ is the *mapping class group* of F. If $X \to S$ is an oriented fiber bundle with fiber F, then analogous to the $\mathrm{MCG}(F)$-covering $\mathcal{R}(X/S)$ of S we construct an $\mathrm{MCG}^+(F)$-covering $\mathcal{P}(X/S)$ of S, whose fiber over a point $s \in S$ consists of the homotopy classes of orientation-preserving homeomorphisms $X_s \to F$.

In particular, if $f : \mathcal{C} \to S$ is a family of genus g curves, then f is also a fiber bundle whose fiber is the compact oriented surface Σ_g of genus g, and Grothendieck calls the $\mathrm{MCG}^+(\Sigma_g)$-covering $\mathcal{P}(\mathcal{C}/S) \to S$ the *Teichmüller covering* of S. The topological space $\mathcal{P}(\mathcal{C}/S)$ obtains the structure of a complex manifold: it is the unique structure for which $\mathcal{P}(\mathcal{C}/S) \to S$ is locally an isomorphism. A *Teichmüller structure* on the family f is a section of the Teichmüller covering $\mathcal{P}(\mathcal{C}/S) \to S$. In other words: giving a Teichmüller structure on f is equivalent to giving a homotopy class of a homeomorphism $\Sigma_g \xrightarrow{\sim} \mathcal{C}_s$ for each $s \in S$, such that these classes 'vary continuously' with s.

Adding Teichmüller structures *rigidifies* the genus g curves we are working with. More precisely: families of genus g curves with Teichmüller structure do not admit nontrivial automorphisms. From this Grothendieck then deduces that the functor

$$S \mapsto \{\text{families of genus } g \text{ curves } f : \mathcal{C} \to S \text{ with Teichmüller structure}\}/\cong$$

is representable. Let $\mathcal{T}_g$ be a representing object; we hence obtain a universal family $\mathcal{X}_g \to \mathcal{T}_g$ of genus g curves with Teichmüller structure. Grothendieck, moreover, remarks that $\mathcal{T}_g$ is homeomorphic to a ball.

Notice, moreover, that there is a natural action of the mapping class group $\Gamma_g = \mathrm{MCG}^+(\Sigma_g)$ on the Teichmüller space $\mathcal{T}_g$. The set of orbits of this action is in bijective correspondence with the set of isomorphism classes of genus g curves. We may therefore view the quotient $M_g = \mathcal{T}_g/\Gamma_g$ as the moduli space of genus g curves. The quotient M_g, however, does not obtain the structure of a complex manifold.

In the remainder of this section we will finish the proof of Proposition 2.2.2. We will be using the following lemma to construct an nontrivial isotrivial family of genus g curves.

Lemma 2.3.1. Let Y and F be topological spaces, and let G be a discrete group. Assume that G acts from the left on Y and F, and assume moreover that the action of G on Y is a covering space action (as defined in [Ful95, Section 1.3]). Note that the actions of G on Y and F induce a G-action on $Y \times F$. Define $S = Y/G$ and $X = (Y \times F)/G$. Then the quotient map $p : Y \to S$ is a covering, and the induced map

$$f : X \to S$$

is a fiber bundle with fiber F. If, moreover, Y is path-connected and the homomorphism $G \to \mathrm{MCG}(F)$ induced by the G-action on F is nontrivial, then the fiber bundle f is nontrivial.

Proof. As the action of G on Y is a covering space action, the quotient map $Y \to Y/G = S$ is a covering map. Moreover, it is straightforward to prove that f is a fiber bundle with fiber F.

Fix points $y \in Y$, $z \in F$, and set $s = p(y)$ and let $x \in X$ be the image of (y, z) under the quotient map $Y \times F \to X$. Note that the composition

$$F \xrightarrow{\sim} \{y\} \times F \hookrightarrow Y \times F \twoheadrightarrow X$$

induces a homeomorphism $F \xrightarrow{\sim} X_s = f^{-1}(s)$; we denote the inverse of this homeomorphism by $\varphi : X_s \xrightarrow{\sim} F$.

The monodromy representation of the pointed G-covering $p : (Y, y) \to (S, s)$ induces a homomorphism

$$\rho : \pi_1(S, s) \to G$$

(c.f. [Ful95, §14a]); it is uniquely determined by the property that $\rho(\alpha) \cdot y = y * \alpha$, where $*$ denotes the monodromy right action of $\pi_1(S, s)$ on the fiber Y_s.

Likewise, the fiber bundle f induces a $\mathrm{MCG}(F)$-covering $\mathcal{R}(X/S) \to S$. The homeomorphism φ induces a point in $\mathcal{R}(X/S)$ over s, and the monodromy representation yields a homomorphism

$$\rho' : \pi_1(S, s) \to \mathrm{MCG}(F).$$

It is now a routine exercise to prove that these two monodromy representations are compatible in the following sense. The action of G on F induces a homomorphism $G \to \mathrm{MCG}(F)$, and the following diagram is commutative:

$$\begin{tikzcd}
\pi_1(S, s) \arrow[rr, "\rho"] \arrow[dr, "\rho'"'] & & G \arrow[dl] \\
& \mathrm{MCG}(F) &
\end{tikzcd}$$

Assume that f is a trivial fiber bundle. Then $\mathcal{R}(X/S) \to S$ is a trivial covering, and the monodromy representation $\pi_1(S, s) \to \mathrm{MCG}(F)$ is trivial. If moreover Y

is path-connected then the monodromy representation $\pi_1(S, s) \to G$ is surjective, which implies that the homomorphism $G \to \mathrm{MCG}(F)$ must be trivial. $\square$

Proof of Proposition 2.2.2 with $g > 0$. Assume that $g > 0$. Let C denote the (hyper)elliptic curve of genus g given by the equation

$$y^2 = \prod_{\lambda=0}^{2g+1} (x - \lambda).$$

Consider the involution $\sigma : C \to C$ given by $(x, y) \mapsto (x, -y)$. Note that σ induces a $\mathbb{Z}$-action on C:

$$\mathbb{Z} \to \mathrm{Aut}(C) : 1 \mapsto \sigma$$

and hence a homomorphism $\mathbb{Z} \to \mathrm{MCG}(C)$ that maps 1 to the class of σ. We claim that the class $[\sigma] \in \mathrm{MCG}(C)$ is nontrivial. In that case, we can consider the $\mathbb{Z}$-covering

$$f : \mathbb{C} \to \mathbb{C}^\times : z \mapsto \exp(2\pi i z),$$

and use Lemma 2.3.1 to construct a nontrivial isotrivial family of genus g curves with fiber C, finishing the proof of Proposition 2.2.2. Note that since f and σ are holomorphic the family obtained from 2.3.1 is a holomorphic fiber bundle.

The involution σ acts as multiplication with -1 on the first (singular) homology group $H_1(C) \cong \mathbb{Z}^{2g}$. Indeed, the group $H_1(C)$ is generated by classes of the form $[\gamma_1 - \gamma_2]$, where γ_1, γ_2 are the two lifts of a path in $\mathbb{P}^1$ between two branch points of the morphism $C \to \mathbb{P}^1 : (x, y) \mapsto x$. The involution σ, then, permutes γ_1 and γ_2, and therefore acts as multiplication by -1 on these classes and hence the whole group. In particular the action of σ on $H_1(C)$ is nontrivial. The automorphism σ, therefore, is not homotopic to the identity id_C, and its class in $\mathrm{MCG}(C)$ is nontrivial. $\square$

2.4 Stacks

As we have seen in Section 2.2 there is no fine moduli space for genus g curves. The reason is the existence of nontrivial automorphisms that we can exploit to 'twist' trivial families into nontrivial isotrivial families. This is a common reason for nonexistence of a fine moduli space for many types of families. We can fix the problem in multiple ways.

One way is to impose extra structure on the objects we classify, as we have seen in Section 2.3. Adding Teichmüller structures to our curves annihilates any nontrivial automorphisms, and a fine moduli space for curves with Teichmüller structure exists.

Another way to circumvent the nonexistence of a fine moduli space for genus g curves is by enlarging our category of complex manifolds by introducing *stacks* over the category of complex manifolds, as was done by Deligne and Mumford [DM69]. For an introduction to stacks we refer to [Fan01], a more thorough treatment is given in [FGI+05]. We also refer to [BX11] and [Hei05] for a treatment of stacks in the context of manifolds.

Roughly speaking, a *stack* (over **CMan**) is a category $\mathcal{M}$ equipped with a functor $F : \mathcal{M} \to \textbf{CMan}$ that allows base changes, gluing of isomorphisms, and gluing of objects.

For instance, consider the category $\mathcal{M}_g$, whose objects are families $f : \mathcal{C} \to S$ of genus g curves, and whose morphisms $(f' : \mathcal{C}' \to S') \to (f : \mathcal{C} \to S)$ are cartesian diagrams

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, "f"] \\
S' \arrow[r] & S.
\end{tikzcd}$$

Moreover, consider the functor $F : \mathcal{M}_g \to \textbf{CMan}$ that maps a family $f : \mathcal{C} \to S$ to its base S.

We observe the following properties:

- If $f : \mathcal{C} \to S$ is a family of genus g curves, and $h : S' \to S$ is any morphism of complex manifolds, then we can take the *base change* $\mathcal{C} \times_S S' \to S'$ of f along h, and this is again a family of genus g curves;
- We can *glue isomorphisms* of families. Let S be a complex manifold with an open covering $S = \bigcup_{i \in I} S_i$, and let $f : \mathcal{C} \to S$ and $f' : \mathcal{C}' \to S$ be families of genus g curves. If we are given isomorphisms between the restrictions of f and f' to S_i for each $i \in I$, and these isomorphisms are compatible on overlaps, then we may glue them to obtain an isomorphism between the families f and f'.
- We can *glue objects* of $\mathcal{M}_g$. If we are given a complex manifold S, an open covering $S = \bigcup_{i \in I} S_i$, for each $i \in I$ a family $f_i : \mathcal{C}_i \to S_i$ of genus g curves, and appropriate gluing data, then we can glue these families together to obtain a family $\mathcal{C} \to S$ of genus g curves.

These three properties ensure that the category $\mathcal{M}_g$ with the functor $\mathcal{M}_g \to \textbf{CMan}$ is a stack.

Definition 2.4.1. The stack $\mathcal{M}_g$ is the *stack of (families of) genus g curves.*

We can view complex manifolds as stacks, too, as the following example demonstrates.

Example 2.4.2. Let S be a complex manifold. Consider the category $[S]$. Objects of $[S]$ are morphisms $f : T \to S$ of complex manifolds. Morphisms $(f : T \to S) \to (f' : T' \to S)$ in $[S]$ are morphisms of complex manifolds $g : T \to T'$ that satisfy $f' \circ g = f$. We fix the functor $[S] \to \textbf{CMan}$ that maps an object $(f : T \to S)$ of $[S]$ to the complex manifold T. This functor gives $[S]$ the structure of a stack.
Let S' be another complex manifold. If $f : S' \to S$ is a morphism of complex manifolds, then composition with f yields a functor $[f] : [S'] \to [S]$, and this

functor is a morphism of stacks: the diagram

$$\begin{tikzcd}
{[S']} \arrow[rr, "{[f]}"] \arrow[dr] & & {[S]} \arrow[dl] \\
& \mathbf{CMan} &
\end{tikzcd}$$

is commutative.
Conversely, if $F : [S'] \to [S]$ is a morphism of stacks, then $F(\mathrm{id}_{S'})$ is a morphism $S' \to S$ of complex manifolds.
One checks that these two operations are inverses, and we therefore see that morphisms of complex manifolds $S' \to S$ correspond bijectively with morphisms of stacks $[S'] \to [S]$.
We will often identify a complex manifold S with its associated stack $[S]$.

Stacks form a 2-category. This means that the morphisms between any two stacks form a category rather than a set. In other words: the category of stacks consists of objects, morphisms, and morphisms between morphisms (which are called 2-morphisms).

Let $F : \mathcal{M} \to$ **CMan** be a stack and let S be a complex manifold. We denote by $\mathcal{M}_S$ the subcategory of $\mathcal{M}$ whose objects are those objects x of $\mathcal{M}$ that satisfy $F(x) = S$, and whose morphisms are those morphisms f of $\mathcal{M}$ that satisfy $F(f) = \mathrm{id}_S$. The 2-Yoneda lemma [SP, Tag 004B] states that there is an equivalence of categories

$$\mathbf{Hom}([S], \mathcal{M}) \xrightarrow{\sim} \mathcal{M}_S$$

given by $F \mapsto F(\mathrm{id}_S)$.

Consider the stack $\mathcal{M}_g$ of families of genus g curves, and let S be a complex manifold. Then $(\mathcal{M}_g)_S$ is the category of genus g curves over S, and the 2-Yoneda lemma gives us an equivalence of categories

$$\mathbf{Hom}([S], \mathcal{M}_g) \xrightarrow{\sim} (\mathcal{M}_g)_S.$$

So morphisms $[S] \to \mathcal{M}_g$ induce families of genus g curves over S. An inverse of Yoneda's equivalence is found as follows: to a family $f : \mathcal{C} \to S$ of genus g curves we associate the functor $\Phi_f : [S] \to \mathcal{M}_g$ given by $(T \to S) \mapsto (f_T : \mathcal{C} \times_S T \to T)$.

Example 2.4.3. Let $\mathcal{C}_g$ be the category whose objects are pairs (f, σ) where $f : \mathcal{C} \to S$ is a family of genus g curves and $\sigma : S \to \mathcal{C}$ is a section of f. Morphisms $(f', \sigma') \to (f, \sigma)$ in $\mathcal{C}_g$ are cartesian diagrams of the form

$$\begin{tikzcd}
\mathcal{C}' \arrow[r, "h'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, "f"] \\
S' \arrow[r, "h"] & S
\end{tikzcd}$$

such that $h' \circ \sigma' = \sigma \circ h$. As families with sections are well-behaved with respect to base changes and gluing, it follows that the functor

$$\mathcal{C}_g \to \mathbf{CMan} : (f : \mathcal{C} \to S, \sigma : S \to \mathcal{C}) \mapsto S$$

gives $\mathcal{C}_g$ the structure of a stack over **CMan**. There is a morphism of stacks $p : \mathcal{C}_g \to \mathcal{M}_g$ that forgets the sections.

Let $f : \mathcal{C} \to S$ be a family of genus g curves. The 2-Yoneda lemma implies that f corresponds to a morphism of stacks $\Phi_f : [S] \to \mathcal{M}_g$. We obtain a morphism $\Psi_f : [\mathcal{C}] \to \mathcal{C}_g$ as follows. An object of $[\mathcal{C}]$ is a morphism $g : T \to \mathcal{C}$ of complex manifolds. The functor Ψ_f then maps g to the family $\mathcal{C} \times_S T \to T$ with the section $(g, \mathrm{id}_T) : T \to \mathcal{C} \times_S T$. We obtain a diagram of stacks

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "\Psi_f"] \arrow[d, "f"'] & \mathcal{C}_g \arrow[d, "p"] \\
S \arrow[r, "\Phi_f"'] & \mathcal{M}_g
\end{tikzcd}$$

and this diagram 2-commutes: there is a 2-isomorphism between the two induced morphisms $\mathcal{C} \to \mathcal{M}_g$. In fact, the diagram induces a representation of the fiber product $S \times_{\mathcal{M}_g} \mathcal{C}_g$ by $\mathcal{C}$. We see that the morphism of stacks $p : \mathcal{C}_g \to \mathcal{M}_g$ behaves like a universal family of genus g curves.

Definition 2.4.4. The *universal family of genus g curves* is the morphism of stacks $p : \mathcal{C}_g \to \mathcal{M}_g$ defined in Example 2.4.3.

Recall that a morphism of stacks $f : \mathcal{X} \to \mathcal{S}$ is *representable* if for each complex manifold and each morphism of stacks $S \to \mathcal{S}$ the fiber product $\mathcal{X} \times_{\mathcal{S}} S$ is again representable by a complex manifold. Equivalently, for each morphism of stacks $\Phi : S \to \mathcal{S}$ there exists a 2-cartesian diagram of the form

$$\begin{tikzcd}
X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\
S \arrow[r, "\Phi"'] & \mathcal{S}
\end{tikzcd}$$

where X is a complex manifold. We say that the morphism $\mathcal{X} \to \mathcal{S}$ is a *submersion* if it is representable and for each cartesian diagram of the above form the morphism of complex manifolds $X \to S$ is a submersion. Analogously, any property of morphisms of complex manifolds that is stable under base change can be generalized to morphisms of stacks. It follows from the discussion in Example 2.4.3 that the universal family of genus g curves $\mathcal{C}_g \to \mathcal{M}_g$ is, indeed, a family of genus g curves.

Example 2.4.5. This example generalizes Example 2.4.3. Let $r \geq 0$ be an integer, and define the category $\mathcal{C}_g^r$ as follows. Objects of $\mathcal{C}_g^r$ are tuples $(f, \sigma_1, \ldots, \sigma_r)$ where f is a family of genus g curves, and $\sigma_1, \ldots, \sigma_r$ are sections of f. Morphisms are cartesian diagrams of families compatible with the sections. The functor $\mathcal{C}_g^r \to$ **CMan** that maps a tuple $(f, \sigma_1, \ldots, \sigma_r)$ to the base of f gives $\mathcal{C}_g^r$ the structure of a stack.
Let $f : \mathcal{C} \to S$ be a family of curves. Let $\mathcal{C}^r$ denote the r-fold fiber product

$$\mathcal{C}^r = \mathcal{C} \times_S \cdots \times_S \mathcal{C}.$$

and for $i = 1, \ldots, r$ let $p_i : \mathcal{C}^r \to \mathcal{C}$ denote the projection onto the ith coordinate. Then f induces a morphism of stacks $\Psi_f^r : [\mathcal{C}^r] \to \mathcal{C}_g^r$ as follows. An object of $[\mathcal{C}^r]$ is a morphism of manifolds $g : T \to \mathcal{C}^r$. Such a morphism induces a family $f_T : \mathcal{C} \times_S T \to T$, together with r sections σ_i given by $\sigma_i = (p_i \circ g, \mathrm{id}_T) : T \to \mathcal{C} \times_S T$. The functor Ψ_f^r maps g to the object $(f_T, \sigma_1, \ldots, \sigma_r)$ of $\mathcal{C}_g^r$. Moreover, the morphism Ψ_f^r, together with the morphism $\mathcal{C}^r \to S$ induced by f, gives rise to a representation by $\mathcal{C}^r$ of the fiber product $\mathcal{C}_g^r \times_{\mathcal{M}_g} S$:

$$\begin{tikzcd}
\mathcal{C}^r \arrow[r, "\Psi_f^r"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_g^r \arrow[d] \\
S \arrow[r, "\Phi_f"'] & \mathcal{M}_g
\end{tikzcd}$$

Here the morphism $\mathcal{C}_g^r \to \mathcal{M}_g$ simply forgets all sections.

The stack $\mathcal{C}_g^r$ defined in Example 2.4.5 is the r-fold fiber product

$$\mathcal{C}_g^r = \mathcal{C}_g \times_{\mathcal{M}_g} \cdots \times_{\mathcal{M}_g} \mathcal{C}_g.$$

Note that $\mathcal{C}_g^1 = \mathcal{C}_g$, and $\mathcal{C}_g^0 = \mathcal{M}_g$.

Let $f : \mathcal{C} \to S$ be a family of genus g curves. For each integer $r \geq 0$ denote by $\mathcal{C}^r$ the r-fold fiber product

$$\mathcal{C}^r = \mathcal{C} \times_S \cdots \times_S \mathcal{C}.$$

Let $r, s \geq 0$ be integers, and let $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ be a map of sets. Then we define a morphism f^ϕ of complex manifolds:

$$f^\phi : \mathcal{C}^r \to \mathcal{C}^s : (x_1, \ldots, x_r) \mapsto (x_{\phi(1)}, \ldots, x_{\phi(s)}).$$

In other words, f^ϕ permutes, forgets, and repeats coordinates of the fiber product $\mathcal{C}^r$. Note that, if $s = 0$, then f^ϕ is the morphism $\mathcal{C}^r \to S$ induced by f.

This construction can be generalized to the universal family $f : \mathcal{C}_g \to \mathcal{M}_g$ as follows. To ϕ we associate a functor $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$:

$$f^\phi : (f, \sigma_1, \ldots, \sigma_r) \mapsto (f, \sigma_{\phi(1)}, \ldots, \sigma_{\phi(s)}).$$

Morphisms in $\mathcal{C}_g^r$ and $\mathcal{C}_g^s$ are cartesian diagrams of families; these are left in place by the functor f^ϕ. It is easy to verify that f^ϕ is a morphism of stacks.

Definition 2.4.6. A *tautological map* is a morphism of stacks of the form $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$, with $r, s \geq 0$ integers and ϕ a map $\{1, \ldots, s\} \to \{1, \ldots, r\}$.

Let $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ be a tautological map associated to a map $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$. Moreover, let $h : \mathcal{C} \to S$ be a family of curves of genus g. Then the diagram

$$\begin{tikzcd}
\mathcal{C}^r \arrow[r, "\Psi_h^r"] \arrow[d, "h^\phi"'] \arrow[dr, phantom, "\square"] & \mathcal{C}_g^r \arrow[d, "f^\phi"] \\
\mathcal{C}^s \arrow[r, "\Psi_h^s"'] & \mathcal{C}_g^s
\end{tikzcd} \tag{2.4.7}$$

is cartesian.

Let $f : \mathcal{C} \to S$ be a family of genus g curves, let $r, s, t, u \geq 0$ be integers, and consider the following commutative diagram of sets and the associated commutative diagram of complex manifolds:

$$\begin{tikzcd}
\{1, \ldots, t\} & \{1, \ldots, r\} \arrow[l, "\psi"'] \\
\{1, \ldots, u\} \arrow[u, "\eta"] & \{1, \ldots, s\} \arrow[l, "\chi"] \arrow[u, "\phi"']
\end{tikzcd} \qquad \begin{tikzcd}
\mathcal{C}^t \arrow[r, "f^\psi"] \arrow[d, "f^\eta"'] & \mathcal{C}^r \arrow[d, "f^\phi"] \\
\mathcal{C}^u \arrow[r, "f^\chi"'] & \mathcal{C}^s
\end{tikzcd}$$

Using the Yoneda lemma it is straightforward to show that if the leftmost diagram is a pushout diagram in the category of sets, then the rightmost diagram is a cartesian diagram. Similarly, if the leftmost diagram is a pushout diagram, the associated diagram of tautological maps between stacks

$$\begin{tikzcd}
\mathcal{C}_g^t \arrow[r, "f^\psi"] \arrow[d, "f^\eta"'] & \mathcal{C}_g^r \arrow[d, "f^\phi"] \\
\mathcal{C}_g^u \arrow[r, "f^\chi"'] & \mathcal{C}_g^s
\end{tikzcd}$$

is cartesian.

Lemma 2.4.8. If $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ is injective, then for each family $f : \mathcal{C} \to S$ of genus g curves the induced morphism $f^\phi : \mathcal{C}^r \to \mathcal{C}^s$ is a submersion. Likewise, if ϕ is injective, then the associated tautological map $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ is a submersion.

Proof. Assume, first, that $s = 0$. Let $f : \mathcal{C} \to S$ be a family of genus g curves. Now f^ϕ is the morphism $\mathcal{C}^r \to S$. As submersions are stable under compositions and base changes, the morphism $\mathcal{C}^r \to S$ is a submersion.

As $s = 0$ we have $\mathcal{C}_g^s = \mathcal{M}_g$. We wish to prove that the morphism $\mathcal{C}_g^r \to \mathcal{M}_g$ is a submersion. Let S be a complex manifold and let $\Phi : S \to \mathcal{M}_g$ be a morphism. Then Φ corresponds to a family $f : \mathcal{C} \to S$ of genus g curves, and the fiber product

$\mathcal{C}_g^r \times_{\mathcal{M}_g} S$ is represented by $\mathcal{C}^r$. The induced morphism of complex manifolds $\mathcal{C}^r \to S$ is the tautological morphism associated to ϕ. As we have seen, this morphism is a submersion of complex manifolds. We may therefore conclude that the tautological map $\mathcal{C}_g^r \to \mathcal{M}_g$ is a submersion.

More generally, let $s \geq 0$ be any integer, and choose an injective map $\eta : \{1, \ldots, r-s\} \to \{1, \ldots, r\}$ whose image is disjoint from the image of ϕ. Let $f : \mathcal{C} \to S$ be a family of genus g curves. We obtain a pushout diagram of sets, and an associated cartesian diagram of complex manifolds:

$$\begin{tikzcd}
\{1, \ldots, r\} & \{1, \ldots, r-s\} \arrow[l, "\psi"'] \\
\{1, \ldots, s\} \arrow[u, "\phi"] & \emptyset \arrow[l] \arrow[u]
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}^r \arrow[r, "f^\psi"] \arrow[d, "f^\phi"'] & \mathcal{C}^{r-s} \arrow[d] \\
\mathcal{C}^s \arrow[r] & S
\end{tikzcd}$$

The morphism $\mathcal{C}^{r-s} \to S$ is a submersion by the first part of this proof, so f^ϕ must be a submersion, too, as submersions are stable under base change.

Analogously, the tautological map $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ can be written as the base change of the submersion $\mathcal{C}_g^{r-s} \to \mathcal{M}_g$ by some tautological map $\mathcal{C}_g^s \to \mathcal{M}_g$, and therefore f^ϕ is a submersion. ☐

Remark 2.4.9. As we already saw in Section 2.3 the Teichmüller space $\mathcal{T}_g$ is closely related to the moduli space of genus g curves. The mapping class group $\Gamma_g = \mathrm{MCG}^+(\Sigma_g)$ of the compact oriented genus g surface Σ_g acts on $\mathcal{T}_g$, and the points in the quotient $\mathcal{T}_g/\Gamma_g$ are in bijective correspondence with isomorphism classes of genus g curves. This quotient, however, does not admit the structure of a complex manifold.

Instead of looking at the topological quotient $\mathcal{T}_g/\Gamma_g$, one could consider the *quotient stack*

$$[\mathcal{T}_g/\Gamma_g],$$

which is defined (in a more general setting of a complex Lie group acting on a complex manifold) as follows. Objects of $[\mathcal{T}_g/\Gamma_g]$ are pairs of morphisms $(P \to S, P \to \mathcal{T}_g)$, where the morphism $P \to S$ is a Γ_g-covering and the morphism $P \to \mathcal{T}_g$ is Γ_g-equivariant. Morphisms are cartesian diagrams of Γ_g-coverings compatible with the equivariant morphisms to $\mathcal{T}_g$. The functor mapping $(P \to S, P \to \mathcal{T}_g) \mapsto S$ gives $[\mathcal{T}_g/\Gamma_g]$ the structure of a stack.

Suppose, now, that $f : \mathcal{C} \to S$ is a family of genus g curves. Recall from Section 2.3 that we obtain a Γ_g-covering $\mathcal{P}(\mathcal{C}/S) \to S$. Points of $\mathcal{P}(\mathcal{C}/S)$ over $s \in S$ are Teichmüller structures on $\mathcal{C}_s$, so we obtain a canonical morphism

$$\mathcal{P}(\mathcal{C}/S) \to \mathcal{T}_g,$$

and this morphism is clearly Γ_g-equivariant.

We hence obtain a canonical morphism of stacks

$$\mathcal{M}_g \to [\mathcal{T}_g/\Gamma_g],$$

We leave it to the reader to verify that this is an isomorphism of stacks and to construct an inverse.

2.5 Differential forms on stacks

2.5.1 Differential forms

Differential forms on complex manifolds can be pulled back along morphisms of manifolds. Moreover, it is possible to glue differential forms along open coverings. As differential forms are well-behaved with respect to pullbacks and gluing, it makes sense to construct a stack of differential forms.

Indeed, let us consider the category A^* whose objects are simply differential forms on *any* complex manifold. If η and ω are some differential forms on complex manifolds T and S, respectively, then the morphisms $\eta \to \omega$ in A^* are precisely those morphisms $f : T \to S$ of the underlying manifolds for which $f^*\omega = \eta$. We consider the functor $A^* \to$ **CMan** that maps a differential form to its underlying complex manifold. Note that this functor is faithful. It is not difficult to verify that this functor makes A^* a stack over **CMan**.

Let S be a complex manifold, and let $\Phi : [S] \to A^*$ be a morphism of stacks. Then $\Phi(\mathrm{id}_S)$ is a differential form on S. Conversely, given a differential form ω on S we can define a morphism $\Phi_\omega : [S] \to A^*$ of stacks that maps a morphism $f : T \to S$ of complex manifolds (that is, an object of $[S]$) to the differential form $f^*\omega$ on T. These constructions are inverses; we see therefore that differential forms on S correspond one-to-one with morphisms of stacks $[S] \to A^*$. This legitimizes the following definition.

Definition 2.5.1. Let $\mathcal{X}$ be a stack over **CMan**. A *differential form on* $\mathcal{X}$ is a morphism of stacks $\mathcal{X} \to A^*$.

Notice that, by the above discussion, differential forms on a complex manifold S correspond canonically to differential forms on the underlying stack $[S]$. In other words: differential forms on stacks generalize differential forms on complex manifolds. From now on, we may identify the differential forms on a complex manifold S with the differential forms on the associated stack $[S]$.

Let $\mathcal{X}$ be a stack, and denote by $\pi : \mathcal{X} \to$ **CMan** the corresponding functor. As the functor $A^* \to$ **CMan** is faithful, any morphism of stacks $\mathcal{X} \to A^*$ over **CMan** is uniquely determined by its action on the objects of $\mathcal{X}$. Giving a differential form ω on $\mathcal{X}$ is, therefore, equivalent to giving for each object x of $\mathcal{X}$ a differential form $\omega(x)$ on the complex manifold $\pi(x)$, such that for each morphism $f : x \to y$ in $\mathcal{X}$ we have the equality $\pi(f)^*\omega(y) = \omega(x)$ of differential forms on $\pi(x)$.

Recall that stacks form a 2-category, so morphisms between two stacks do not form a set but a category. For an arbitrary stack $\mathcal{X}$ we obtain a category (and not a set) $A^*(\mathcal{X})$ of differential forms on $\mathcal{X}$. Fortunately, it is easy to verify that there are no 2-morphisms between two differential forms on any given stack, apart from identity morphisms. So $A^*(\mathcal{X})$ is a discrete category; differential forms on $\mathcal{X}$ form a class. If $\mathcal{X} = [S]$ is the stack associated to a complex manifold, then the objects of $A^*([S])$ are in bijective correspondence with differential forms on S. We can therefore view the discrete category $A^*([S])$ as a set by identifying its objects with the elements of $A^*(S)$.

2.5.2 Pullbacks

Suppose that $f : S' \to S$ is a morphism of complex manifolds, and let $[f] : [S'] \to [S]$ denote the associated morphism of stacks. Let $\omega \in A^*(S)$ be a differential form on S. Recall that differential forms on a complex manifold correspond bijectively to differential forms on the associated stack. If ω corresponds to the morphism of stacks $\Phi_\omega : [S] \to A^*$, then the pullback $f^*\omega$ corresponds to the composition $\Phi_\omega \circ [f] : [S'] \to A^*$. So it makes sense to define pullbacks of differential forms along morphisms of stacks as follows.

Definition 2.5.2. Let $f : \mathcal{X}' \to \mathcal{X}$ be a morphism of stacks. Given a differential form $\omega : \mathcal{X} \to A^*$ we define the *pullback* of ω along f to be the differential form $f^*\omega := \omega \circ f : \mathcal{X}' \to A^*$ on $\mathcal{X}'$.

This definition generalizes the definition of pullbacks of differential forms on complex manifolds.

Example 2.5.3. The following example allows us to switch seamlessly between evaluating differential forms on objects of stacks and taking pullbacks of differential forms.
Let $\mathcal{X}$ be a stack with a differential form $\omega : \mathcal{X} \to A^*$. Let X be any complex manifold. Recall the 2-Yoneda equivalence

$$\mathbf{Hom}(X, \mathcal{X}) \xrightarrow{\sim} \mathcal{X}_X : \Phi \mapsto \Phi(\mathrm{id}_X).$$

Let $\Phi : X \to \mathcal{X}$ be a morphism of stacks and let x be an object of $\mathcal{X}_X$. If $\Phi(\mathrm{id}_X) \cong x$ in $\mathcal{X}_X$, then we have an equality

$$\Phi^*\omega = \omega(x) \in A^*(X).$$

The following observation is useful when working with differential forms on stacks. Let $f, g : \mathcal{X}' \to \mathcal{X}$ be two morphisms of stacks, and let $\omega : \mathcal{X} \to A^*$ be a differential form on $\mathcal{X}$. Assume that there exists a 2-isomorphism between f and g. Then the compositions $\omega \circ f$ and $\omega \circ g : \mathcal{X}' \to A^*$ are 2-isomorphic as well. As there are no nontrivial 2-isomorphisms between differential forms, it follows that $f^*\omega = g^*\omega$.

The following is a generalization of Lemma 1.1.6.

Lemma 2.5.4. Let $f : \mathcal{X} \to \mathcal{S}$ be a submersion of stacks. Then the functor $f^* : A^*(\mathcal{S}) \to A^*(\mathcal{X})$ is injective.

Proof. Let ω and η be two differential forms on $\mathcal{S}$ such that $f^*\omega = f^*\eta$. In order to prove that $\omega = \eta$, it suffices to show that these functors evaluate equally on all objects of $\mathcal{S}$. Let s be an object of $\mathcal{S}$ over the complex manifold S. By using the

2-Yoneda lemma, we can construct a 2-cartesian diagram of the form

$$\begin{tikzcd}X \arrow[r, "\Psi"] \arrow[d, "f_S"] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\ S \arrow[r, "\Phi"] & \mathcal{S}\end{tikzcd}$$

such that X is a complex manifold, and such that $\Phi(\mathrm{id}_S) \cong s$ in $\mathcal{S}$. We now have the following equality of differential forms on X:

$$f_S^*(\omega(s)) = f_S^*\Phi^*\omega = \Psi^* f^*\omega = \Psi^* f^*\eta = f_S^*\Phi^*\eta = f_S^*(\eta(s)).$$

As f_S is a submersion, we deduce from Lemma 1.1.6 that $\omega(s) = \eta(s)$. □

2.5.3 Fiber integrals

Now, let us generalize taking fiber integrals to the setting of stacks. Recall that a morphism $\mathcal{X} \to \mathcal{S}$ of stacks is a *submersion* if it is representable and a submersion. That is, for each complex manifold S and each morphism $\Phi : S \to \mathcal{S}$ there exists a 2-cartesian diagram of the form

$$\begin{tikzcd}X \arrow[r, "\Psi"] \arrow[d, "f_S"] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\ S \arrow[r, "\Phi"] & \mathcal{S}\end{tikzcd} \tag{2.5.5}$$

where X is a complex manifold, and the morphism $f_S : X \to S$ is a submersion of complex manifolds.

We must first generalize the notion of differential forms with proper support over the base of a submersion to the setting of stacks. By Proposition 1.3.14 this property is stable under base change, and therefore it makes sense to generalize it as follows.

Definition 2.5.6. Let $f : \mathcal{X} \to \mathcal{S}$ be a submersion of stacks, and let ω be a differential form on $\mathcal{X}$. We say that ω has *proper support* over $\mathcal{S}$ if for each 2-cartesian diagram of the form 2.5.5 the pullback $\Psi^*\omega \in A^*(X)$ has proper support over S.

It follows from Proposition 1.3.14 that for each submersion $f : X \to S$ of complex manifolds, and each differential form $\omega \in A^*(X)$, the form ω has proper support over S if and only if the corresponding differential form on the stack $[X]$ has proper support over $[S]$.

Now, let us generalize the fiber integral operator along submersions of complex manifolds to the setting of stacks. Let $f : \mathcal{X} \to \mathcal{S}$ be a submersion of stacks, and let ω be a differential form on $\mathcal{X}$ with proper support over $\mathcal{S}$. Moreover, we denote the (implicitly given) functor $\mathcal{S} \to$ **CMan** by π. We will construct a differential form $\int_f \omega$ on $\mathcal{S}$ as follows. Let s be any object of $\mathcal{S}$, and let $S = \pi(s)$. The functor $\int_f \omega$ should assign to s a differential form on S. By applying the 2-Yoneda lemma, we can construct a cartesian diagram of the form 2.5.5, such that $\Phi(\mathrm{id}_S) \cong s$ in $\mathcal{S}_S$. We obtain a differential form on S by pulling back ω along the morphism $\Psi : X \to \mathcal{X}$, and then taking the fiber integral of the resulting form along the submersion $f_S : X \to S$.

The following lemma implies that the resulting form $(\int_f \omega)_s$ on S does not depend on any choices. Moreover, one can show using this lemma that the above construction indeed defines a differential form on $\mathcal{S}$.

Lemma 2.5.7. Let $F : \mathcal{X} \to \mathcal{S}$ be a submersion of stacks. Assume we have two 2-cartesian diagrams of stacks

$$\begin{tikzcd}
X_1 \arrow[r, "\Psi_1"] \arrow[d, "f_1"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "F"] \\
S \arrow[r, "\Phi_1"'] & \mathcal{S}
\end{tikzcd}
\qquad\qquad
\begin{tikzcd}
X_2 \arrow[r, "\Psi_2"] \arrow[d, "f_2"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "F"] \\
S \arrow[r, "\Phi_2"'] & \mathcal{S}
\end{tikzcd}$$

where X_1, X_2, S are complex manifolds, and assume that there exists a 2-isomorphism $\Phi_1 \Rightarrow \Phi_2$, or, equivalently, that $\Phi_1(\mathrm{id}_S) \cong \Phi_2(\mathrm{id}_S)$ in $\mathcal{S}_S$. Then for each differential form ω on $\mathcal{X}$ with proper support over $\mathcal{S}$ we have

$$\int_{f_1} \Psi_1^* \omega = \int_{f_2} \Psi_2^* \omega \in A^*(S).$$

Proof. Any 2-isomorphism $\Phi_1 \Rightarrow \Phi_2$ induces a morphism $u : X_1 \to X_2$ that makes the following cube-shaped diagram 2-commute:

$$\begin{tikzcd}
X_1 \arrow[rr, "\Psi_1"] \arrow[dr, "u"] \arrow[dd, "f_1"'] & & \mathcal{X} \arrow[dr, "="] \arrow[dd, "F"'] & \\
& X_2 \arrow[rr, "\Psi_2"] \arrow[dd, "f_2"] & & \mathcal{X} \arrow[dd, "F"] \\
S \arrow[rr, "\Phi_1"] \arrow[dr, "="'] & & \mathcal{S} \arrow[dr, "="] & \\
& S \arrow[rr, "\Phi_2"] & & \mathcal{S}
\end{tikzcd}$$

See [SP, Tag 02XA]. Of this cube, the front, back, and rightmost face are 2-cartesian, so the same holds for the leftmost face, which is therefore a cartesian square of complex manifolds. By chasing through the above diagram we find the equality

$$\int_{f_1} \Psi_1^* \omega = \int_{f_1} u^* \Psi_2^* \omega = \mathrm{id}_S^* \int_{f_2} \Psi_2^* \omega = \int_{f_2} \Psi_2^* \omega,$$

where the middle equality follows from Proposition 1.3.14. $\square$

The defining property of the fiber integral can therefore be given as follows.

Definition 2.5.8. Let $f : \mathcal{X} \to \mathcal{S}$ be a submersion of stacks, and let $\omega : \mathcal{X} \to A^*$ be a differential form on $\mathcal{X}$ with proper support over $\mathcal{S}$. The *fiber integral* of ω along f is the unique differential form $\int_f \omega : \mathcal{S} \to A^*$ on $\mathcal{S}$ that satisfies the following property: for each cartesian diagram of the form 2.5.5, one has:

$$\Phi^* \left(\int_f \omega \right) = \left(\int_f \omega \right) (\Phi(\mathrm{id}_S)) = \int_{f_S} \Psi^* \omega \in A^*(S).$$

The properties of the fiber integral, as listed in Section 1.3, can be generalized immediately to the setting of stacks. For example, the base change formula 1.3.14 generalizes as follows.

Proposition 2.5.9 (Base change formula for stacks)**.** Consider a 2-cartesian diagram of stacks

$$\begin{tikzcd}
\mathcal{X}' \arrow[r, "h"] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\
\mathcal{S}' \arrow[r, "g"'] & \mathcal{S}
\end{tikzcd}$$

where f and f' are submersions. If ω is a differential form on $\mathcal{X}$ that has proper support over $\mathcal{S}$, then $h^*\omega$ has proper support over $\mathcal{S}'$, and the following equality holds:

$$g^* \left(\int_f \omega \right) = \int_{f'} h^* \omega.$$

Proof. Suppose that we are given a 2-cartesian diagram of the form

$$\begin{tikzcd}
X \arrow[r, "\Psi"] \arrow[d, "f_S"] \arrow[dr, phantom, "\square"] & \mathcal{X}' \arrow[d, "f'"] \\
S \arrow[r, "\Phi"] & \mathcal{S}'
\end{tikzcd}$$

where X and S are complex manifolds. Then the following diagram is 2-cartesian, too:

$$\begin{tikzcd}
X \arrow[r, "h \circ \Psi"] \arrow[d, "f_S"] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\
S \arrow[r, "g \circ \Phi"] & \mathcal{S}
\end{tikzcd}$$

By assumption the pullback $(h \circ \Psi)^*\omega = \Psi^* h^* \omega$ has proper support over S. We deduce that $h^*\omega$ has proper support over $\mathcal{S}'$.

Let s' be any object of $\mathcal{S}'$, and let S be its image under the functor $\mathcal{S}' \to$ **CMan**. Define $s = g(s')$. By the 2-Yoneda lemma, we can construct 2-cartesian diagrams as in the first part of this proof, such that $\Phi(\mathrm{id}_S) \cong s'$ in $\mathcal{S}'_S$. We then have, by definition of the fiber integral along f':

$$\left(\int_{f'} h^* \omega \right) (s') = \int_{f_S} \Psi^* h^* \omega \in A^*(S').$$

Note, moreover, that we have an isomorphism $(g \circ \Phi)(\mathrm{id}_S) \cong g(s') = s$ in $\mathcal{S}_S$. Applying the definition of the fiber integral along f now yields

$$\left(g^* \int_f \omega\right)(s') = \left(\int_f \omega\right)(g(s')) = \left(\int_f \omega\right)(s) = \int_{f_{s'}} (h \circ \Psi)^* \omega \in A^*(S).$$

We find that the two differential forms evaluate equally on objects of $\mathcal{S}'$, so they are equal. $\square$

2.5.4 Differential forms on moduli stacks

Let $p : \mathcal{C}_g \to \mathcal{M}_g$ be the universal family of genus g curves. In this thesis we are mostly interested in differential forms on the moduli stacks $\mathcal{C}_g^r = \mathcal{C}_g \times_{\mathcal{M}_g} \cdots \times_{\mathcal{M}_g} \mathcal{C}_g$ for $r \geq 0$, where $\mathcal{C}_g^0 = \mathcal{M}_g$ and $\mathcal{C}_g^1 = \mathcal{C}_g$. In this section we will see that we can often pretend that these stacks are honest complex manifolds, when it comes to studying differential forms on them. In particular, we will be able to view pullbacks and fiber integrals along morphisms between these stacks in an intuitive way.

Let $f : \mathcal{C} \to S$ be a family of genus g curves. Recall that f corresponds to a morphism $\Phi_f : S \to \mathcal{M}_g$, and that for all $r \geq 0$ we have morphisms $\Psi_f^r : \mathcal{C}^r \to \mathcal{C}_g^r$ that make the following diagram cartesian:

$$\begin{tikzcd}
\mathcal{C}^r \arrow[r, "\Psi_f^r"] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{C}_g^r \arrow[d, "p"] \\
S \arrow[r, "\Phi_f"] & \mathcal{M}_g
\end{tikzcd}$$

Proposition 2.5.10. Let $r \geq 0$ be an integer. Let ω be a differential form on $\mathcal{C}_g^r$. For every family $f : \mathcal{C} \to S$ denote by ω_f the differential form on $\mathcal{C}^r$ obtained by pulling back ω along the canonical morphism $\Psi_f^r : \mathcal{C}^r \to \mathcal{C}_g^r$. The forms ω_f are compatible with base change: if we have a cartesian diagram

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, "f"] \\
S' \arrow[r] & S
\end{tikzcd}$$

where f and f' are families of genus g curves, then the pullback of ω_f along the induced morphism $\mathcal{C}'^r \to \mathcal{C}^r$ equals $\omega_{f'}$.

Conversely, if we are given a differential form $\omega_f \in A^*(\mathcal{C}^r)$ for each family $f : \mathcal{C} \to S$ of genus g curves, and these forms are compatible with base change, then there is a unique differential form ω on $\mathcal{C}_g^r$ such that $(\Psi_f^r)^*\omega = \omega_f$ for each family $f : \mathcal{C} \to S$ of genus g curves.

In other words: differential forms on $\mathcal{M}_g$ are differential forms that occur universally on the bases of families of genus g curves, differential forms on $\mathcal{C}_g$ are differential forms that occur universally on the sources of families of genus g

curves, and analogous statements hold for differential forms on $\mathcal{C}_g^r$ for $r \geq 2$. We will prove Proposition 2.5.10 later in this section.

The proposition also implies that taking pullbacks and fiber integrals along tautological maps works 'as expected'. Indeed, let $r, s \geq 0$ be integers, let $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ be a map, and let $p^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ be the associated tautological map. Recall that we have for each family $f : \mathcal{C} \to S$ an induced 2-cartesian diagram of stacks (2.4.7):

$$\begin{tikzcd}
\mathcal{C}^r \arrow[r, "\Psi_f^r"] \arrow[d, "f^\phi"'] \arrow[dr, phantom, "\square"] & \mathcal{C}_g^r \arrow[d, "p^\phi"] \\
\mathcal{C}^s \arrow[r, "\Psi_f^s"'] & \mathcal{C}_g^s
\end{tikzcd}$$

Let ω be a differential form on $\mathcal{C}_g^s$. By the proposition, ω induces for each family $f : \mathcal{C} \to S$ a differential form $\omega_f := (\Psi_f^s)^* \omega$ on $\mathcal{C}^s$. Likewise, the pullback $\omega' := (p^\phi)^* \omega$ on $\mathcal{C}_g^r$ associates to each family $f : \mathcal{C} \to S$ a differential form $\omega'_f := (\Psi_f^r)^* \omega'$ on $\mathcal{C}^r$. But as the above diagram 2-commutes, we simply find that $\omega'_f = (f^\phi)^* \omega_f$ for each family f. So, roughly speaking, under the correspondences of Proposition 2.5.10, taking pullbacks of differential forms along tautological maps works 'as expected'.

An analogous statement can be made for fiber integrals. Suppose ϕ is injective, so p^ϕ is a submersion. Let ω be a differential form on $\mathcal{C}_g^r$, and set $\omega' := \int_{p^\phi} \omega$. By the base change formula we have $(\Psi_f^s)^* \omega' = \int_{f^\phi} \Psi_f^r \omega$. Therefore the fiber integral is compatible with the correspondences of Proposition 2.5.10, too.

These observations will allow us to pretend that moduli stacks behave like honest complex manifolds in Chapter 4 when we are working with differential forms on these stacks.

Another observation we should make is the following. Assume that $g \geq 2$. Recall from Section 2.3 that the stack $\mathcal{T}_g$ of families of genus g curves with Teichmüller structure is representable by a complex manifold. We have, moreover, a morphism of stacks $\mathcal{T}_g \to \mathcal{M}_g$. This is a covering map. Indeed, let S be any complex manifold, and let $\Phi : S \to \mathcal{M}_g$ be any morphism. Then Φ corresponds to a family of curves $f : \mathcal{C} \to S$. Consider the covering $\mathcal{P}(\mathcal{C}/S) \to S$ as defined in Section 2.3, and notice that the base change $\mathcal{P}(\mathcal{C}/S) \times_S \mathcal{C} \to \mathcal{P}(\mathcal{C}/S)$ is a family of genus g curves with a *canonical* Teichmüller structure. We hence obtain a canonical morphism $\mathcal{P}(\mathcal{C}/S) \to \mathcal{T}_g$. The following diagram is 2-commutative:

$$\begin{tikzcd}
\mathcal{P}(\mathcal{C}/S) \arrow[r] \arrow[d] & \mathcal{T}_g \arrow[d] \\
S \arrow[r, "\Phi"] & \mathcal{M}_g
\end{tikzcd}$$

In fact, this diagram is 2-cartesian: it induces a representation of the fiber product $\mathcal{T}_g \times_{\mathcal{M}_g} S$ by the complex manifold $\mathcal{P}(\mathcal{C}/S)$. It follows that the morphism $\mathcal{T}_g \to \mathcal{M}_g$ is representable, and a covering, and in particular a submersion. This implies that the pullback operator

$$A^*(\mathcal{M}_g) \to A^*(\mathcal{T}_g)$$

is injective. We may therefore view $A^*(\mathcal{M}_g)$ as a *subset* of $A^*(\mathcal{T}_g)$.

Analogously, the induced morphism $\mathcal{X}_g^r \to \mathcal{C}_g^r$ is a covering map for all $r \geq 1$, and we obtain inclusions

$$A^*(\mathcal{C}_g^r) \to A^*(\mathcal{X}_g^r).$$

We will finish this section by proving Proposition 2.5.10. We will use the following lemmas.

Lemma 2.5.11. Recall that the objects of the stack $\mathcal{C}_g$ are pairs (f, σ), where f is a family of genus g curves and σ is a section of f. Let $\omega : \mathcal{C}_g \to A^*$ be a differential form. Let $f : \mathcal{C} \to S$ be a family of genus g curves, and let $\Psi_f : \mathcal{C} \to \mathcal{C}_g$ be the canonical morphism. Then we have an equality of differential forms

$$\Psi_f^* \omega = \omega(p_1, \Delta) \in A^*(\mathcal{C}),$$

where $p_1 : \mathcal{C}^2 \to \mathcal{C}$ is the projection and $\Delta : \mathcal{C} \to \mathcal{C}^2$ is its diagonal section.
If, moreover, $\sigma : S \to \mathcal{C}$ is a section of f, then

$$\omega(f, \sigma) = \sigma^* \Psi_f^* \omega \in A^*(S).$$

Proof. The morphism of stacks $\Psi_f : [\mathcal{C}] \to \mathcal{C}_g$ maps the canonical object $\mathrm{id}_{\mathcal{C}}$ of $[\mathcal{C}]$ to the pair (p_1, Δ), which proves the first statement.

For the second statement, consider the cartesian diagram with sections

$$\begin{tikzcd}
\mathcal{C} \arrow[r, "(\sigma f, \mathrm{id}_{\mathcal{C}})"] \arrow[d, "f"] \arrow[dr, phantom, "\square"] & \mathcal{C}^2 \arrow[d, "p_1"'] \\
S \arrow[r, "\sigma"] \arrow[u, bend left=60, "\sigma"] & \mathcal{C} \arrow[u, bend right=60, "\Delta"']
\end{tikzcd}$$

As ω is a functor, we find:

$$\sigma^* \Psi_f^* \omega = \sigma^* \omega(p_1, \Delta) = \omega(f, \sigma). \qquad \square$$

Lemma 2.5.12. Assume we are given for each family $f : \mathcal{C} \to S$ of genus g curves a differential form $\omega_f \in A^*(\mathcal{C})$, compatible with base change. Then for each family $f : \mathcal{C} \to S$ we have an equality

$$\Delta^* \omega_{p_1} = \omega_f$$

where $p_1 : \mathcal{C}^2 \to \mathcal{C}$ is the projection and $\Delta : \mathcal{C} \to \mathcal{C}^2$ is its diagonal section.

Proof. Consider the cartesian diagram

$$\begin{tikzcd}
\mathcal{C}^2 \arrow[r, "p_2"] \arrow[d, "p_1"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, "f"] \\
\mathcal{C} \arrow[r, "f"] & S
\end{tikzcd}$$

We find: $p_2^*\omega_f = \omega_{p_1}$, and pulling this equality back along the diagonal gives the desired result. $\square$

Proof of Proposition 2.5.10. Verifying Proposition 2.5.10 is straightforward if $r = 0$.

Suppose, now, that $r = 1$. Let $\omega : \mathcal{C}_g \to A^*$ be a differential form, and for each family $f : \mathcal{C} \to S$ set

$$\omega_f := \Psi_f^*\omega \in A^*(\mathcal{C}).$$

If we have a cartesian diagram as in the statement of the proposition, then the following diagram is 2-commutative:

$$\begin{tikzcd}
\mathcal{C}' \arrow[rrd, bend left, "\Psi_{f'}"] \arrow[rd] \arrow[dd, "f'"] & & \\
& \mathcal{C} \arrow[r, "\Psi_f"] \arrow[dd, "f"] & \mathcal{C}_g \arrow[dd, "p"] \\
S' \arrow[rd] \arrow[rrd, bend left, "\Phi_{f'}"] & & \\
& S \arrow[r, "\Phi_f"] & \mathcal{M}_g
\end{tikzcd}$$

We therefore find that the pullback of ω_f along the morphism $\mathcal{C}' \to \mathcal{C}$ equals $\omega_{f'}$.

Conversely, suppose that we have for each family $f : \mathcal{C} \to S$ a differential form $\omega_f \in A^*(\mathcal{C})$, compatible with base change. We then construct a functor $\omega : \mathcal{C}_g \to A^*$ as follows: ω sends a pair (f, σ), with f a family and σ a section, to the differential form $\sigma^*\omega_f \in A^*(S)$. As the forms ω_f are compatible with base change, this defines a morphism of stacks, so we obtain a differential form ω on $\mathcal{C}_g$.

Lemmas 2.5.11 and 2.5.12 now imply that the two constructions we described above are inverses.

The proof for $r \geq 2$ is very similar and hence omitted. $\square$

Remark 2.5.13. Recall from Remark 2.4.9 that we may view $\mathcal{M}_g$ as the quotient stack $[\mathcal{T}_g/\Gamma_g]$, where Γ_g is the mapping class group of the compact oriented genus g surface Σ_g that acts on the Teichmüller space $\mathcal{T}_g$. Moreover we have a canonical submersion $\mathcal{T}_g \to \mathcal{M}_g$, and the corresponding pullback operator gives an inclusion $A^*(\mathcal{M}_g) \to A^*(\mathcal{T}_g)$. The image of this inclusion consists of the Γ_g-invariant forms on $\mathcal{T}_g$. This gives us yet another way of thinking about differential forms on $\mathcal{M}_g$.

2.6 Hermitian vector bundles on moduli spaces of curves

Vector bundles on complex manifolds are well-behaved: we can take pullbacks of vector bundles, glue vector bundles on open coverings, and glue isomorphisms of vector bundles. It therefore makes sense to construct a stack of vector bundles on complex manifolds, as follows. The *stack of (rank n) vector bundles* $\mathcal{V}_n$ has as its objects holomorphic vector bundles $E \to S$ of rank n, and its morphisms are pullback diagrams of vector bundles

$$\begin{tikzcd}
E' \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E \arrow[d] \\
S' \arrow[r] & S.
\end{tikzcd}$$

The functor $\mathcal{V}_n \to \mathbf{CMan}$ sends a vector bundle $E \to S$ to its base space S.

If $\mathcal{X}$ is any other stack over **CMan**, then a *vector bundle (of rank n)* on $\mathcal{X}$ is a morphism of stacks $\mathcal{X} \to \mathcal{V}_n$. We obtain a category $\mathcal{V}_n(\mathcal{X}) = \mathbf{Hom}(\mathcal{X}, \mathcal{V}_n)$ of rank n vector bundles on $\mathcal{X}$. If S is a complex manifold, the 2-Yoneda lemma gives an equivalence of categories between the category $\mathcal{V}_n([S])$ of vector bundles on the stack $[S]$ and the category $(\mathcal{V}_n)_S$ of vector bundles on the complex manifold S. Note that, unlike in the setting of differential forms, this is not a bijection but 'merely' an equivalence of categories. This is to be expected: the pullback of a vector bundle along a morphism of manifolds is only defined up to a unique isomorphism.

If $f : \mathcal{X}' \to \mathcal{X}$ is a morphism of stacks, and $E : \mathcal{X} \to \mathcal{V}_n$ a rank n vector bundle on $\mathcal{X}$, we can define the *pullback* f^*E to be the rank n vector bundle $E \circ f : \mathcal{X}' \to \mathcal{V}_n$ on $\mathcal{X}'$. We obtain a pullback functor $f^* : \mathcal{V}_n(\mathcal{X}) \to \mathcal{V}_n(\mathcal{X}')$.

Analogously, one can define the *stack of hermitian vector bundles (of rank n)* $\overline{\mathcal{V}}_n$ in a similar way: its objects are hermitian vector bundles of rank n, and its morphisms are base change diagrams that induce isometries on all fibers. A *hermitian vector bundle of rank n* on a stack $\mathcal{X}$ is then a morphism of stacks $\mathcal{X} \to \overline{\mathcal{V}}_n$. For each morphism $f : \mathcal{X}' \to \mathcal{X}$ we obtain a pullback functor $f^* : \overline{\mathcal{V}}_n(\mathcal{X}) \to \overline{\mathcal{V}}_n(\mathcal{X}')$.

Analogous to Proposition 2.5.10 we have:

Example 2.6.1. The category $\mathcal{V}_n(\mathcal{M}_g)$ of rank n vector bundles on $\mathcal{M}_g$ has as its objects functors $\mathcal{M}_g \to \mathcal{V}_n$ over **CMan**. That is: a rank n vector bundle E on $\mathcal{M}_g$ assigns to each family $f : \mathcal{C} \to S$ of genus g curves a rank n vector bundle $E(f) \to S$, and to each cartesian square of the form

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, "f"] \\
S' \arrow[r, "h"] & S
\end{tikzcd}$$

a pullback diagram

$$\begin{tikzcd}
E(f') \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & E(f) \arrow[d] \\
S' \arrow[r, "h"] & S,
\end{tikzcd}$$

under the obvious compatibility criterion with respect to compositions of cartesian squares.
Let E_1, E_2 be two rank n vector bundles on $\mathcal{M}_g$. A morphism of vector bundles $\phi : E_1 \to E_2$ is a morphism of functors over **CMan**. That is: ϕ assigns to each family of genus g curves a morphism $\phi(f) : E_1(f) \to E_2(f)$ of vector bundles over S, such that for each cartesian diagram of curves as above the following induced diagram is commutative:

$$\begin{tikzcd}
E_1(f') \arrow[r, "\phi(f')"] \arrow[d] \arrow[drr] & E_2(f') \arrow[dl] \arrow[drr] & & \\
S' \arrow[drr, "h"] & & E_1(f) \arrow[r, "\phi(f)"] \arrow[d] & E_2(f) \arrow[dl] \\
& & S &
\end{tikzcd}$$

Hermitian vector bundles on $\mathcal{M}_g$ can be described analogously.

Analogous to the situation with differential forms, one can show that giving a (hermitian) vector bundle on $\mathcal{C}_g^r$ is equivalent to assigning to every family $f : \mathcal{C} \to S$ of genus g curves a (hermitian) vector bundle on the r-fold fiber product $\mathcal{C}^r$ over S, under the corresponding base change compatibility criterion.

Example 2.6.2. Suppose that $f : \mathcal{C} \to S$ is a family of curves. On the diagonal bundle $O(\Delta)$ we have constructed a canonical hermitian metric in Chapter 1.
This construction is stable under base change, so universally we obtain a hermitian line bundle $O(\Delta)$ on $\mathcal{C}_g^2$.

Example 2.6.3. Let $f : \mathcal{C} \to S$ be a family of curves. Recall that we have a canonical isomorphism

$$\Delta^* O(\Delta) \xrightarrow{\sim} \omega_f^{\otimes -1}$$

and this canonical isomorphism induces a canonical metric on the relative cotangent bundle ω.
This construction, too, is compatible with base change. We therefore obtain a canonical hermitian line bundle $\omega = \omega_{\mathcal{C}_g/\mathcal{M}_g}$ on the universal family $\mathcal{C}_g$ of genus g curves.

The *first Chern form* is a differential form c_1 on the stack of hermitian line bundles $\overline{\mathcal{V}}_1$, defined as follows. The functor $c_1 : \overline{\mathcal{V}}_1 \to A^*$ takes a hermitian line bundle $L \to S$ and maps it to the differential form $c_1(L) \in A^*(S)$, where $c_1(L)$ denotes the first Chern form of L on the complex manifold S. This gives a well-defined functor as taking first Chern forms on complex manifolds commutes with taking pullbacks. It follows that for each stack $\mathcal{X}$ and each hermitian line bundle $\mathcal{L}$ on $\mathcal{X}$ we can take the first Chern form of $\mathcal{L}$ by composing with c_1 to obtain a differential form $c_1(\mathcal{L})$ on $\mathcal{X}$. Notice that this construction generalizes taking the first Chern form of a hermitian line bundle on a complex manifold.

Next, we will generalize the construction of the Deligne pairing to the setting of stacks. Suppose that $f : \mathcal{X} \to \mathcal{S}$ is a morphism of stacks, and suppose moreover that f is a family of curves. Let $L, M : \mathcal{X} \to \overline{\mathcal{V}}_1$ be two hermitian line bundles on $\mathcal{C}_g$. We will define a hermitian line bundle $\langle L, M\rangle : \mathcal{S} \to \overline{\mathcal{V}}_1$ as follows. For each object s of $\mathcal{S}$ over the complex manifold S choose a cartesian diagram of the form

$$\begin{tikzcd} X \arrow[r, "\Psi_s"] \arrow[d, "f_s"] \arrow[dr, phantom, "\square"] & \mathcal{X} \arrow[d, "f"] \\ S \arrow[r, "\Phi_s"] & \mathcal{S}. \end{tikzcd}$$

where X is a complex manifold, and Φ_s is such that $\Phi_s(\mathrm{id}_S) \cong s$ in $\mathcal{S}_S$. Then $L(\Psi_s(\mathrm{id}_X))$ and $M(\Psi_s(\mathrm{id}_X))$ are line bundles on X. Taking the Deligne pairing of these line bundles along the family f_s of curves then yields a line bundle $\langle L(\Psi_s(\mathrm{id}_X)), M(\Psi_s(\mathrm{id}_X))\rangle$ on S. The functor $\langle L, M\rangle : \mathcal{S} \to \overline{\mathcal{V}}_1$ maps s to this line bundle. A morphism in $\mathcal{S}$ is mapped to the canonically induced pullback diagram of corresponding line bundles. Note that the functor $\langle L, M\rangle : \mathcal{S} \to \overline{\mathcal{V}}_1$ does depend on choices, and is only determined up to 2-isomorphism. In other words: the Deligne pairing of L and M is a line bundle on $\mathcal{S}$, defined up to isomorphism.

It follows immediately that Proposition 1.4.13 generalizes to the setting of stacks: we have the following equality of differential forms on $\mathcal{S}$:

$$c_1(\langle L, M\rangle) = \int_f c_1(L) \wedge c_1(M).$$

In particular, the Deligne pairing along tautological submersions $\mathcal{C}_g^{r+1} \to \mathcal{C}_g^r$ behaves as expected.

> **Example 2.6.4.** Consider the diagonal bundle $O(\Delta)$ on $\mathcal{C}_g^2$ and the relative cotangent bundle $\omega = \omega_{\mathcal{C}_g/\mathcal{M}_g}$ on $\mathcal{C}_g$ with their canonical metrics. We have an equality of differential forms on $\mathcal{C}_g$:
>
> $$\int_{p_1:\mathcal{C}_g^2\to\mathcal{C}_g} c_1(O(\Delta))^2 = c_1(\langle O(\Delta), O(\Delta)\rangle) = c_1(\Delta^* O(\Delta)) = -c_1(\omega).$$

2.7 The universal Jacobian bundle

The *Jacobian* of the universal family $p : \mathcal{C}_g \to \mathcal{M}_g$ is a stack whose objects are pairs (f, σ) where $f : \mathcal{C} \to S$ is a family of genus g curves and $\sigma : S \to \mathcal{J}_{\mathcal{C}/S}$ is a section of the Jacobian family $\mathcal{J}_{\mathcal{C}/S}$ associated to f. Morphisms $(f', \sigma') \to (f, \sigma)$ in $\mathcal{J}_g$ are cartesian diagrams

$$\begin{tikzcd} \mathcal{C}' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, "f"] \\ S' \arrow[r] & S \end{tikzcd}$$

such that the induced diagram

$$\begin{tikzcd}
\mathcal{J}_{\mathcal{C}'/S'} \arrow[r] & \mathcal{J}_{\mathcal{C}/S} \\
S' \arrow[u, "\sigma'"] \arrow[r] & S \arrow[u, "\sigma"']
\end{tikzcd}$$

is commutative. The functor $\mathcal{J}_g \to \mathbf{CMan}$ that maps a pair (f, σ) to the base of f gives $\mathcal{J}_g$ the structure of a stack. Forgetting sections yields a canonical morphism of stacks $\mathcal{J}_g \to \mathcal{M}_g$.

Recall that every family $f : \mathcal{C} \to S$ of genus g curves gives rise to a morphism of stacks $\Phi_f : S \to \mathcal{M}_g$. It is straightforward to show that the fiber product of stacks $S \times_{\mathcal{M}_g} \mathcal{J}_g$ is then represented by the relative Jacobian $\mathcal{J}_{\mathcal{C}/S}$ of f. More precisely: there is a natural morphism $\mathcal{J}_{\mathcal{C}/S} \to \mathcal{J}_g$, and the following diagram of stacks is 2-cartesian:

$$\begin{tikzcd}
\mathcal{J}_{\mathcal{C}/S} \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & \mathcal{J}_g \arrow[d] \\
S \arrow[r, "\Phi_f"] & \mathcal{M}_g
\end{tikzcd}$$

It follows that the morphism of stacks $\mathcal{J}_g \to \mathcal{M}_g$ is a family of complex tori.

The following analogue to Proposition 2.5.10 is easily seen to hold:

Proposition 2.7.1. Let ω be a differential form on $\mathcal{J}_g$. For each family $f : \mathcal{C} \to S$ of genus g curves let $\mathcal{J}_f \to S$ denote the relative Jacobian family of f, and let $\omega_f \in A^*(\mathcal{J}_f)$ denote the pullback of ω along the induced morphism of stacks $\mathcal{J}_f \to \mathcal{J}_g$. The forms ω_f are compatible with base change: for each cartesian diagram

$$\begin{tikzcd}
\mathcal{C}' \arrow[r] \arrow[d, "f'"] \arrow[dr, phantom, "\square"] & \mathcal{C} \arrow[d, "f"] \\
S' \arrow[r] & S
\end{tikzcd}$$

with f and f' families of genus g curves, the pullback of ω_f along the induced morphism $\mathcal{J}_{f'} \to \mathcal{J}_f$ equals $\omega_{f'}$.
Conversely, if we are given a differential form $\omega_f \in A^*(\mathcal{J}_f)$ for each family f of genus g curves, satisfying the necessary compatibility conditions under pullbacks, then there is a unique differential form ω on $\mathcal{J}_g$ such that the pullback of ω along the canonical morphism $\mathcal{J}_f \to \mathcal{J}_g$ equals ω_f for each family f of genus g curves. □

Similarly, (hermitian) vector bundles on $\mathcal{J}_g$ can be viewed as (hermitian) vector bundles that occur universally on the relative Jacobians of all families of genus g curves. For instance, we obtain the canonical hermitian line bundle $\mathcal{B}$ on the universal Jacobian bundle $\mathcal{J}_g$. This allows us to generalize the results from Section 1.4 to the universal setting. For instance, we have canonical morphisms of stacks

$$\delta : \mathcal{C}_g^2 \to \mathcal{J}_g \quad \text{and} \quad \kappa : \mathcal{C}_g \to \mathcal{J}_g.$$

Here δ maps a family $f : \mathcal{C} \to S$ with sections $\sigma_1, \sigma_2 : S \to \mathcal{C}$ to the pair (f, σ) in $\mathcal{J}_g$, with σ the section

$$\sigma : S \to \mathcal{J}_f : s \mapsto [O(\sigma_2(s) - \sigma_1(s))] \in \mathrm{Jac}(\mathcal{C}_s).$$

Likewise, the morphism κ maps a family $f : \mathcal{C} \to S$ with section σ to the pair (f, σ) in $\mathcal{J}_g$, where σ is the section

$$\sigma : S \to \mathcal{J}_f : s \mapsto [O((2g-2)\sigma(s)) \otimes \omega^{\otimes -1}] \in \mathrm{Jac}(\mathcal{C}_s).$$

We then have canonical isometries

$$\delta^* \mathcal{B}^{\otimes -1} \xrightarrow{\sim} p_1^* \omega^{\otimes -1} \otimes p_2^* \omega^{\otimes -1} \otimes O(\Delta)^{\otimes -2}$$

and

$$\kappa^* \mathcal{B}^{\otimes -1} \xrightarrow{\sim} \omega^{-2g(2g-2)} \otimes p^* \langle \omega, \omega \rangle_p.$$

of hermitian vector bundles on $\mathcal{C}_g^2$ and $\mathcal{C}_g$, respectively.

Chapter 3

Marked graphs

In this chapter we will study r-marked graphs, where $r \geq 0$ is an integer. These are finite graphs of which r of the vertices are labeled with the integers $1, \ldots, r$. We define categories of r-marked graphs, show that these categories have pushouts, and construct pushforward and pullback functors between these categories.

A contracted r-marked graph is an r-marked graph whose vertices have a sufficiently high degree, and any r-marked graph can be turned into a contracted r-marked graph by means of certain contraction operations. It turns out that for each $r \geq 0$ and each $\chi \in \mathbb{Z}$ there are only finitely many isomorphism classes of contracted r-marked graphs of characteristic χ. We will describe an algorithm to compute the number of isomorphism classes.

The reason we are interested in r-marked graphs is that they provide us with a combinatorial framework that can be used to work with tautological differential forms. Fix an integer $g > 1$. In Chapter 4, we will describe a method of assigning to each r-marked graph Γ a tautological differential form $\alpha_\Gamma \in R^*(\mathcal{C}_g^r)$. It turns out that there is an interaction between r-marked graphs and tautological forms on $\mathcal{C}_g^r$, where taking pushouts corresponds to taking wedge products, and taking pushforwards and pullbacks of graphs corresponds to taking pullbacks and fiber integrals of forms.

Moreover, it turns out that *all* tautological differential forms on $\mathcal{C}_g^r$ arise from contracted r-marked graphs. We can give upper bounds to the dimensions of spaces of tautological forms by computing the number of marked graphs. In short, the combinatorial heavy lifting will be done in this chapter, and we use the results from this chapter to bound dimensions of spaces of tautological forms in Chapter 4.

3.1 The category of r-marked graphs

In this thesis, a *graph* is a pair (V, E), consisting of a finite set V of *vertices*, and a finite multiset E of *edges* consisting of unordered pairs (multisets of cardinality 2) of elements of V. If $e \in E$ is an edge, its two elements are called the *endpoints* of e. If these endpoints are the same, we call e a *loop*. The *degree* of a vertex $v \in V$, denoted $\deg v$, is the number of times v occurs as an endpoint of an edge of E; that is: the multiplicity of v in the multiset sum of all edges $e \in E$. In particular, we see that each loop contributes 2 to the degree of the vertex it is based on.

In short, we assume our graphs to be finite and undirected, and our graphs are allowed to have multiple edges and loops. Moreover, our graphs do not necessarily have to be connected.

If $\Gamma = (V, E)$ is a graph, then the *(Euler) characteristic* of Γ is defined as

$$\chi(\Gamma) = |V| - |E|.$$

The Euler characteristic is additive on disjoint unions of graphs.

Let $r \geq 0$ be an integer. An *r-marked graph* (V, E, m) is a graph $\Gamma = (V, E)$ equipped with a *marking* m; that is: an injective map $m : \{1, \ldots, r\} \to V$. So a marked graph can be seen as a graph of which r vertices are labeled $1, \ldots, r$. An *unmarked graph* is a 0-marked graph, which is the same as an 'ordinary' graph.

Let $\Gamma = (V, E, m)$ be an r-marked graph. A vertex $v \in V$ is *marked* if it is in the image of m, and *unmarked* otherwise. We have a partition of V in a subset V_+ of marked vertices and a subset V_- of unmarked vertices.

Let $\Gamma = (V, E, m)$ and $\Gamma' = (V', E', m')$ be two r-marked graphs. A *morphism of r-marked graphs* $f : \Gamma \to \Gamma'$ is a pair of maps $(f_{\mathrm{v}} : V \to V', f_{\mathrm{e}} : E \to E')$, such that f_{v} respects the r-marking (that is: $f_{\mathrm{v}} \circ m = m'$), and such that for each edge $e \in E$ with endpoints v, w, the edge $f_{\mathrm{e}}(e) \in E'$ has endpoints $f_{\mathrm{v}}(v)$ and $f_{\mathrm{v}}(w)$.

We obtain a category $\mathcal{G}_r$ of r-marked graphs. Two r-marked graphs Γ and Γ' are isomorphic if and only if there exists a bijection on vertices that respects the markings of Γ and Γ', such that for each pair of vertices v, w of Γ the number of edges between v and w equals the number of edges between the corresponding vertices of Γ'.

Example 3.1.1. The following two 1-marked graphs are not isomorphic:

Indeed, the marked vertex in the leftmost graph has degree 4, while the marked vertex in the rightmost graph has degree 2.
The corresponding 0-marked graphs, obtained by 'forgetting' the 1-markings, are isomorphic.

The following construction will return in the next sections. Assume that $\Gamma = (V, E)$ is a graph, and let $f : V \to V'$ be a map of finite sets. The *graph induced (from Γ) by f*, notation Γ_f, is the graph (V', E') with set of vertices equal to V', and with edges

$$E' = \{\{f(v_1), f(v_2)\} : \{v_1, v_2\} \in E\}.$$

Notice that in particular we have $|E| = |E'|$.

The characteristic of Γ_f equals

$$\chi(\Gamma_f) = \chi(\Gamma) + |V'| - |V|.$$

If $v' \in V'$ is a vertex in V', its degree is given by:

$$\deg(v') = \sum_{v \in f^{-1}(v')} \deg(v).$$

3.2 Gluing marked graphs

In this section we define a binary operation $\sqcup_r$ on the category of r-marked graphs $\mathcal{G}_r$. It turns out that $\sqcup_r$ is the coproduct in the category $\mathcal{G}_r$. We define the binary operation $\sqcup_r$ on two r-marked graphs Γ, Γ' by gluing their marked vertices pairwise. More precisely, we proceed as follows.

Let $\Gamma = (V, E, m)$ and $\Gamma' = (V', E', m')$ be two r-marked graphs, and let $\Gamma \sqcup \Gamma' = (V \sqcup V', E + E')$ denote the disjoint union of the underlying (unmarked) graphs. Consider the set V'' defined by the pushout diagram

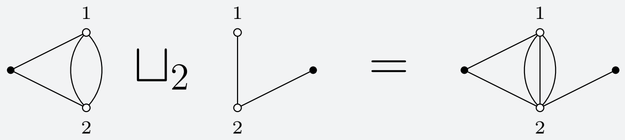
$$\begin{tikzcd}
\{1, \dots, r\} \arrow[r, "m"] \arrow[d, "m'"] \arrow[dr, phantom, "\lrcorner", very near end] & V \arrow[d] \\
V' \arrow[r] & V''.
\end{tikzcd} \tag{3.2.1}$$

In other words, V'' is the set $(V \sqcup V')/\sim$, where $\sim$ is the smallest equivalence relation on $V \sqcup V'$ such that $m(i) \sim m'(i)$ for all $i \in \{1, \dots, r\}$. Note, moreover, that the map $m'' : \{1, \dots, r\} \to V''$ induced by the above diagram is injective, since m and m' are injective.

Definition 3.2.2. The r-marked graph $\Gamma \sqcup_r \Gamma'$ is the graph induced from the disjoint union $\Gamma \sqcup \Gamma'$ by the natural map $V \sqcup V' \to V''$, endowed with the r-marking $m'' : \{1, \dots, r\} \to V''$ obtained from pushout diagram 3.2.1.

In short: we take the disjoint union of Γ and Γ', and identify the vertices in V and V' whose markings are equal.

Example 3.2.3. The following picture illustrates the operation $\sqcup_2$ on $\mathcal{G}_2$.



Suppose that Γ has u unmarked vertices and e edges, and that Γ' has u' unmarked vertices and e' edges. It follows that $\Gamma \sqcup_r \Gamma'$ has $u + u'$ unmarked vertices and $e + e'$ edges. Therefore, the characteristic of $\Gamma \sqcup_r \Gamma'$ is given by

$$\chi(\Gamma \sqcup_r \Gamma') = \chi(\Gamma) + \chi(\Gamma') - r. \tag{3.2.4}$$

The set of vertices of $\Gamma \sqcup_r \Gamma'$ is the pushout of the maps m and m'. It is therefore straightforward to verify the following.

Proposition 3.2.5. For each pair of r-marked graphs Γ_1, Γ_2, the graph $\Gamma_1 \sqcup_r \Gamma_2$ is the coproduct of Γ_1 and Γ_2 in the category $\mathcal{G}_r$. $\square$

We find that the operator $\sqcup_0$ on $\mathcal{G}_0$ is simply the disjoint union. On $\mathcal{G}_1$ the operator $\sqcup_1$ is the wedge sum.

Proposition 3.2.6. The category $\mathcal{G}_r$ has all finite coproducts.

Proof. The graph consisting of r marked vertices, no unmarked vertices, and no edges is the initial object of $\mathcal{G}_r$. As $\mathcal{G}_r$ has an initial object and all binary coproducts, it follows that $\mathcal{G}_r$ has all finite coproducts. $\square$

3.3 Pushforward maps on marked graphs

Let $\phi : \{1, \dots, s\} \to \{1, \dots, r\}$ be a map of sets. We will define a pushforward functor $\phi_* : \mathcal{G}_s \to \mathcal{G}_r$. Given a graph $\Gamma \in \mathcal{G}_s$, the pushforward $\phi_*\Gamma$ is obtained from Γ by replacing the s marked vertices by r marked vertices, as follows.

Let $\Gamma = (V, E, m)$ be an s-marked graph. Consider the pushout diagram (of sets)

$$\begin{tikzcd}
\{1, \dots, s\} \arrow[r, "m"] \arrow[d, "\phi"] & V \arrow[d, "\phi_V"] \\
\{1, \dots, r\} \arrow[r, "m'"'] \arrow[ur, phantom, "\lrcorner", very near end] & V'.
\end{tikzcd} \tag{3.3.1}$$

As m is injective, it follows that m' must be injective.

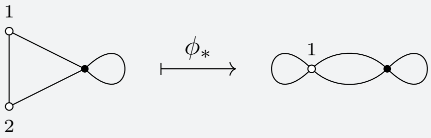
We define $\phi_*\Gamma$ to be the graph (V', E', m'), where (V', E') is the graph induced from (V, E) by ϕ_V, and m' is the map defined in diagram 3.3.1. Notice that ϕ_V then induces a bijection between the unmarked vertices of Γ and $\phi_*\Gamma$.

Alternatively, we can construct the graph $\phi_*\Gamma$ as follows: first, we construct a graph Γ'' by adding r vertices $v''_1, \dots, v''_r$ to Γ. Next, we define the equivalence relation $\sim$ on the set of vertices V'' to be the smallest equivalence relation such that $v_i \sim v''_{\phi(i)}$ for all $i \in \{1, \dots, s\}$, where $v_1, \dots, v_s$ are the marked vertices of Γ. Then $\phi_*\Gamma$ is the graph quotient $\Gamma''/\sim$.

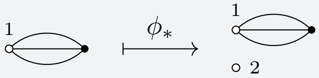
The characteristic of $\phi_*\Gamma$ is given by

$$\chi(\phi_*\Gamma) = \chi(\Gamma) - s + r.$$

Example 3.3.2. Let $\phi : \{1, 2\} \to \{1\}$ be the unique map. The pushforward $\phi_* : \mathcal{G}_2 \to \mathcal{G}_1$ identifies the two marked points of a 2-marked graph. The following picture illustrates taking the pushforward $\phi_*\Gamma$ of a graph $\Gamma \in \mathcal{G}_2$:



Example 3.3.3. Consider the map $\phi : \{1\} \to \{1,2\}$ that maps 1 to 1. The pushforward map ϕ_* adds a second marked vertex to any 1-marked graph Γ. This second marked vertex is not the endpoint of any edge.



Moreover, if $f : \Gamma_1 \to \Gamma_2$ is a morphism of s-marked graphs, we obtain an induced morphism of r-marked graphs $\phi_* f : \phi_*\Gamma_1 \to \phi_*\Gamma_2$, via the universal property of the pushout diagram 3.3.1. We obtain a covariant functor

$$\phi_* : \mathcal{G}_s \to \mathcal{G}_r.$$

It is not hard to see that the pushforward functor is well-behaved with respect to compositions.

Proposition 3.3.4. Let $\phi : \{1,\dots,s\} \to \{1,\dots,r\}$ and $\psi : \{1,\dots,t\} \to \{1,\dots,s\}$ be maps. Then the functors $\phi_*\psi_*$ and $(\phi\psi)_*$ from $\mathcal{G}_t$ to $\mathcal{G}_r$ are naturally isomorphic. ☐

The pushforward operator is compatible with gluing.

Proposition 3.3.5. Let $\phi : \{1,\dots,s\} \to \{1,\dots,r\}$ be a map. Let Γ and Γ' be two s-marked graphs. Then there is a canonical isomorphism of graphs

$$\phi_*(\Gamma \sqcup_s \Gamma') \simeq \phi_*(\Gamma) \sqcup_r \phi_*(\Gamma').$$

Proof. Let V and V' denote the sets of vertices of Γ and Γ', respectively. The set V_1 of vertices of $\phi_*(\Gamma \sqcup_s \Gamma')$ can be obtained by repeatedly taking pushouts:

$$V_1 = (V \sqcup_{\{1,\dots,s\}} V') \sqcup_{\{1,\dots,s\}} \{1,\dots,r\}$$

The same holds for the set V_2 of vertices of $\phi_*(\Gamma) \sqcup_r \phi_*(\Gamma')$:

$$V_2 = (V \sqcup_{\{1,\dots,s\}} \{1,\dots,r\}) \sqcup_{\{1,\dots,r\}} (V' \sqcup_{\{1,\dots,s\}} \{1,\dots,r\}),$$

By the universal property of pushouts these sets are canonically isomorphic. It is straightforward to verify that the canonical isomorphism induces a bijection on edges. ☐

3.4 Pullback maps on marked graphs

The next operation we will consider is a pullback operation. Let

$$\phi : \{1, \dots, s\} \to \{1, \dots, r\}$$

be an *injective* map. Then we define a *pullback* functor

$$\phi^* : \mathcal{G}_r \to \mathcal{G}_s$$

as follows. For any r-marked graph $\Gamma = (V, E, m)$ we simply define $\phi^*\Gamma$ by precomposing the marking m with the injection ϕ:

$$\phi^*(\Gamma) = (V, E, m \circ \phi).$$

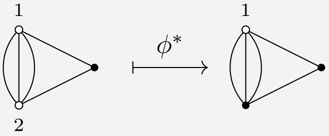
So all the pullback functor does is maybe re-ordering and possibly forgetting some markings of vertices.

It follows that

$$\chi(\phi^*\Gamma) = \chi(\Gamma).$$

If $f : \Gamma_1 \to \Gamma_2$ is a morphism of r-marked graphs, then f induces a morphism $\phi^* f : \phi^*\Gamma_1 \to \phi^*\Gamma_2$ in a natural way. It is straightforward to verify that ϕ^* is a functor from $\mathcal{G}_r \to \mathcal{G}_s$.

Example 3.4.1. Let $\phi : \{1\} \to \{1, 2\}$ be the inclusion. The pullback $\phi^* : \mathcal{G}_2 \to \mathcal{G}_1$ takes a 2-marked graph and turns it into a 1-marked graph by forgetting the marking of the second marked point.



Similarly to the pushforward, it is easy to see that the pullback is well-behaved with respect to compositions.

Proposition 3.4.2. Let $\phi : \{1, \dots, s\} \to \{1, \dots, r\}$ and $\psi : \{1, \dots, t\} \to \{1, \dots, s\}$ be injective maps. Then the functors $\psi^*\phi^*$ and $(\phi\psi)^*$ from $\mathcal{G}_r$ to $\mathcal{G}_t$ are equal. ☐

There is an adjointness between the pushforward and pullback functor. Contrary to what the terms 'pushforward' and 'pullback' might suggest to a geometer, the pushforward functor is *left* adjoint to the pullback. To ease our minds, we recall that the (left adjoint) pushforward functor does pushouts on sets of vertices, and the (right adjoint) pullback functor is a functor that *forgets* some of the markings.

Proposition 3.4.3. If $\phi : \{1, \dots, s\} \to \{1, \dots, r\}$ is an injective map, the pushforward functor $\phi_* : \mathcal{G}_s \to \mathcal{G}_r$ is left adjoint to the pullback functor $\phi^* : \mathcal{G}_r \to \mathcal{G}_s$.

Proof. The statement follows from the universal property of the pushout diagram 3.3.1, together with the fact that the pushforward and pullback maps do not alter the sets of edges. ☐

3.5 The monoid of r-marked graphs

In the previous sections we have defined the categories of marked graphs, showed that these categories have coproducts, and constructed pushforward and pullback functors between these categories. Later on, we will try to classify such r-marked graphs, or rather a specific subset of *contracted* r-marked graphs. In Chapter 4 we will use this classification of r-marked graphs in order to classify tautological differential forms. For these purposes, categories of marked graphs are too big; we only need to find all r-marked graphs up to isomorphism.

Definition 3.5.1. Let $r \geq 0$ be an integer. Then we denote by $\mathrm{G}(r)$ the set of isomorphism classes of r-marked graphs. If $\chi \in \mathbb{Z}$ is another integer, we let $\mathrm{G}(r, \chi) \subseteq \mathrm{G}(r)$ denote the subset of the isomorphism classes of r-marked graphs with characteristic χ. Lastly, if $u \geq 0$ is an integer, the subset $\mathrm{G}(r, \chi, u) \subseteq \mathrm{G}(r, \chi)$ denotes the subset of the classes of the graphs that have u unmarked vertices.

Of course, we need to be a bit careful here and remark that this definition of $\mathrm{G}(r)$ does not yield a set under the ZFC axioms, as almost every element of $\mathrm{G}(r)$ will be a proper class. However, it is straightforward to construct a set S of r-marked graphs such that every r-marked graph is isomorphic to one in S, and we can then view $G(r)$ as the quotient set $S/\cong$.

Given isomorphisms of r-marked graphs $\Gamma_1 \cong \Gamma_2$ and $\Gamma_1' \cong \Gamma_2'$, there exists a natural isomorphism $(\Gamma_1 \sqcup_r \Gamma_1') \cong (\Gamma_2 \sqcup_r \Gamma_2')$. This implies that the operator $\sqcup_r$ defines a binary operation on the set $\mathrm{G}(r)$. We immediately obtain the following proposition.

Proposition 3.5.2. Equipping the set $\mathrm{G}(r)$ with the binary operation $\sqcup_r$ yields a commutative monoid, whose identity element is (the class of) the r-marked graph with no edges and no unmarked vertices. ☐

It follows from Equation 3.2.4 that the map

$$\chi_r : \mathrm{G}(r) \to \mathbb{Z} : \Gamma \mapsto \chi(\Gamma) - r$$

is a homomorphism of monoids.

Suppose, now, that $\phi : \{1, \dots, s\} \to \{1, \dots, r\}$ is a map of sets. This map defines a pushforward functor on marked graphs $\phi_* : \mathcal{G}_s \to \mathcal{G}_r$. This functor induces a map $\phi_* : \mathrm{G}(s) \to \mathrm{G}(r)$. The following proposition follows directly from Proposition 3.3.5.

Proposition 3.5.3. For every map $\phi : \{1, \dots, s\} \to \{1, \dots, r\}$, the map $G(s) \to G(r)$ induced by the functor $\phi_* : \mathcal{G}_s \to \mathcal{G}_r$ is a homomorphism of monoids. $\square$

If ϕ is an injective map, we also have a pullback functor $\phi^* : \mathcal{G}_r \to \mathcal{G}_s$. The induced map on monoids $\mathrm{G}(r) \to \mathrm{G}(s)$, however, is almost never a homomorphism. For instance, consider the identity element $\Gamma \in \mathrm{G}(r)$, which is the r-marked graph with no edges and no unmarked vertices. Then $\phi^*\Gamma$ is an s-marked graph with $(r - s)$ unmarked vertices, so this graph is not the identity element of $\mathrm{G}(s)$ unless $r = s$.

3.6 Contracted graphs

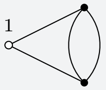
In Chapter 4 we will fix an integer $g > 1$, and associate to any r-marked graph Γ a differential form α_Γ on the moduli stack $\mathcal{C}_g^r$. It will turn out that the form α_Γ is invariant under certain contraction operations on these r-marked graphs. This will allow us to restrict ourselves to studying differential forms associated to graphs which cannot be contracted any further. We will study such graphs in this section.

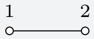
Definition 3.6.1. Let Γ be an r-marked graph. Then Γ is *contracted* if all its unmarked vertices have degree at least 3, and each unmarked vertex of degree 3 is incident to three distinct edges.
We denote by $\mathrm{CG}(r) \subseteq \mathrm{G}(r)$ the subset of isomorphism classes of contracted r-marked graphs.

Equivalently, a marked graph is contracted if each unmarked vertex has degree at least 3, and there are no loops at any of the unmarked vertices of degree 3.

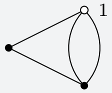
Notice that, by construction of the binary operation $\sqcup_r$, the subset $\mathrm{CG}(r) \subseteq \mathrm{G}(r)$ is in fact a submonoid.

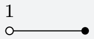
Example 3.6.2. The following marked graphs are contracted.

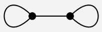




The following marked graphs are not contracted.







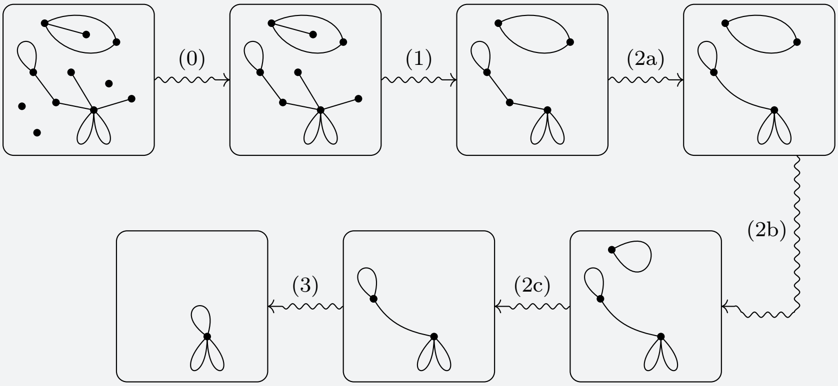
If a graph is not contracted, we can attempt to turn this graph into a contracted graph by altering the problematic vertices.

Definition 3.6.3. Let $\Gamma = (V, E, m)$ be an r-marked graph, and let $v \in V$ be an unmarked vertex such that $\deg(v) \leq 2$, or such that $\deg(v) = 3$ and v is incident to a loop. The graph obtained from Γ by *contracting* v is an r-marked graph Γ' defined by the following operation:

0. If $\deg v = 0$, remove v;
1. If $\deg v = 1$, remove v and the unique edge incident to v;
2. If $\deg v = 2$, *smooth out* the vertex v; that is:
    (a) If v is incident to two distinct edges, whose other endpoints w, w' are distinct, remove v and these two edges, and add an edge between w and w';
    (b) If v is incident to two distinct edges, whose second endpoint is the same vertex w, remove v and these two edges, and add a loop at w;
    (c) If v is incident to a single loop, remove v and this loop;
3. If $\deg v = 3$, and w is the other endpoint of the non-loop edge incident to v, remove v, this edge, and the loop at v, and add a loop at w.

It follows that the vertex set of the graph obtained from Γ by contracting v is equal to $V \setminus \{v\}$.

Example 3.6.4. The following example illustrates all of the graph manipulations listed in the above definition. The graph is unmarked.



If we are given an r-marked graph Γ, we can always reduce Γ to a contracted r-marked graph by applying a finite amount of graph contractions. We hence obtain a map

$$\varrho : \mathrm{G}(r) \to \mathrm{CG}(r),$$

and this map is a retraction of the inclusion $\mathrm{CG}(r) \subseteq \mathrm{G}(r)$. Moreover, as the contraction operations only apply to unmarked vertices, it follows that the contraction map commutes with gluing of r-marked graphs:

$$\varrho(\Gamma_1 \sqcup_r \Gamma_2) = \varrho(\Gamma_1) \sqcup_r \varrho(\Gamma_2).$$

In other words: $\mathrm{CG}(r)$ is a submonoid of $\mathrm{G}(r)$ and the contraction map $\varrho : \mathrm{G}(r) \to \mathrm{CG}(r)$ is a homomorphism of monoids.

3.7 Counting contracted graphs

The number of contracted r-marked graphs is infinite. For instance, for $r = 0$, and $e \geq 2$, we can consider the unmarked graph consisting of 1 vertex and e loops. This example gives us an infinite family of contracted unmarked graphs. However, a finiteness result does hold if we only consider r-marked graphs of a fixed characteristic.

Theorem 3.7.1. Let $r \geq 0$, and $\chi \in \mathbb{Z}$. There are, up to isomorphism, only finitely many contracted r-marked graphs of characteristic χ. These graphs have at most $2(r - \chi)$ unmarked vertices.

Proof. Let Γ be a contracted r-marked graph of characteristic χ, and let u denote its number of unmarked vertices, and e its number of edges. We have:

$$\chi = r + u - e.$$

Moreover, every unmarked vertex has degree at least 3. It follows that

$$2e = \sum_{v \in \Gamma} \deg(v) \geq 3u.$$

After substituting $e = r + u - \chi$, we find:

$$u \leq 2r - 2\chi,$$

and hence

$$e = r + u - \chi \leq 3r - 3\chi.$$

We have obtained upper bounds for the number of vertices and edges of Γ, and a simple combinatorial argument then shows that there can only be finitely many graphs of this form up to isomorphism. □

We let $\mathrm{CG}(r, \chi) \subseteq \mathrm{CG}(r)$ denote the set of equivalence classes of contracted r-marked graphs of characteristic χ. By Theorem 3.7.1 the cardinality of this set is finite. Moreover, we denote by $\mathrm{CG}(r, \chi, u) \subseteq \mathrm{CG}(r, \chi)$ the set of isomorphism classes of graphs with u unmarked vertices. Theorem 3.7.1 yields:

$$\mathrm{CG}(r, \chi) = \bigsqcup_{u=0}^{2(r-\chi)} \mathrm{CG}(r, \chi, u).$$

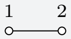
Example 3.7.2. In this example we will compute the set $\mathrm{CG}(r, r-1)$ for all

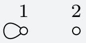
$r \geq 0$. By Theorem 3.7.1 we have:

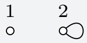
$$\mathrm{CG}(r, r-1) = \mathrm{CG}(r, r-1, 0) \sqcup \mathrm{CG}(r, r-1, 1) \sqcup \mathrm{CG}(r, r-1, 2).$$

Note that for each graph $\Gamma \in \mathrm{CG}(r, r-1, u)$ with r marked vertices, u unmarked vertices, and e edges, one has $\chi(\Gamma) = r + u - e = r - 1$, so $e = u + 1$.

- Every graph $\Gamma \in \mathrm{CG}(r, r-1, 0)$ has no unmarked vertices and one edge (possibly a loop) between two (not necessarily distinct) marked vertices. It follows that there are $\frac{1}{2}r(r+1)$ such graphs.

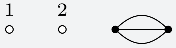






- Every graph $\Gamma \in \mathrm{CG}(r, r-1, 1)$ has two edges and one unmarked vertex. As Γ is contracted and contains only two distinct edges, its unmarked vertex has degree at least 4, so the two edges of Γ must be two loops based at the unmarked vertex. We find that $\mathrm{CG}(r, r-1, 1)$ only has one element.

- Every graph $\Gamma \in \mathrm{CG}(r, r-1, 2)$ has three edges and two unmarked vertices. As each unmarked vertex must have degree ≥ 3 and there are only three edges in Γ, it follows that the degree of both unmarked vertices equals 3, and that they are both incident to each of the three edges of Γ. Therefore $\mathrm{CG}(r, r-1, 2)$ has only one element.



We find that the number of elements of $\mathrm{CG}(r, r-1)$ equals

$$|\mathrm{CG}(r, r-1)| = \tfrac{1}{2}r(r+1) + 1 + 1 = \tfrac{1}{2}r^2 + \tfrac{1}{2}r + 2.$$

In general the set $\mathrm{CG}(r, \chi, u)$ can be computed with an algorithm as follows:

Algorithm 3.7.3. The following (pseudocode) algorithm computes all isomorphism classes of contracted r-marked graphs with u unmarked vertices and characteristic χ.

```
def compute_CG(r,χ,u):
    L = ∅  # list of graphs
    e = r + u - χ  # number of edges
    Γ = (r-marked graph with u unmarked vertices, no edges)
    P = {unordered pairs of vertices of Γ}  # possible edges
    # loop over all possible configurations of edges:
    for E in {multisets of size e with elements from P}:
```

```
        ΓE = Γ equipped with edges from E
        # check if ΓE is contracted, and if ΓE was not found before:
        if ΓE.is_contracted() and ΓE ≇ Γ2 for all Γ2 in L:
            # if so, add Γ to the list
            L.add(ΓE)
    return L
```

Moreover, Theorem 3.7.1 implies that we can compute the set $\mathrm{CG}(r, \chi)$ in finite time by computing $\mathrm{CG}(r, \chi, u)$ for $0 \leq u \leq 2(r-\chi)$ and taking their disjoint union.

Of course, the combinatorial complexity of constructing graphs, checking if they are contracted, and checking if they are isomorphic to any of the graphs we found before, will become worse and worse if r increases and if χ decreases. In Section 3.8 we will show that for all $d \geq 0$ the size of $\mathrm{CG}(r, r-d)$ in terms of r can be expressed as a polynomial of degree $2d$; so if we are given d, we can compute a formula for $|\mathrm{CG}(r, r-d)|$ for all $r \geq 0$ in finite time. In Section 3.9 we will then compute these polynomials for $d \leq 4$ by computing its first values and applying Lagrange interpolation.

3.8 The size of $\mathrm{CG}(r, r-d)$ in terms of r

In this section we will prove that, given an integer $d \geq 0$, one can compute a closed formula for the number of contracted r-marked graphs of characteristic $r-d$ for all $r \geq 0$. This will prove useful in Chapter 4, as it allows us to compute upper bounds for the dimensions of spaces of tautological forms on $\mathcal{C}_g^r$ that do not depend on the genus g.

Theorem 3.8.1. Let $d \in \mathbb{Z}$ be an integer. If d is negative, $\mathrm{CG}(r, r-d)$ is empty for all $r \geq 0$. If $d \geq 0$, then there exists a polynomial $f_d \in \mathbb{Q}[x]$ of degree $2d$ such that

$$|\mathrm{CG}(r, r-d)| = f_d(r) \quad \text{for all } r \geq 0.$$

The leading coefficient of f_d is $1/(2^d \cdot d!)$.

In fact, the following stronger theorem directly implies Theorem 3.8.1.

Theorem 3.8.2. Let $d \in \mathbb{Z}$ and $u \geq 0$ be integers. If $2d-u$ is negative, then $\mathrm{CG}(r, r-d, u)$ is empty for all $r \geq 0$. If $2d-u \geq 0$, then there exists a polynomial $f_{d,u} \in \mathbb{Q}[x]$ of degree at most $2d-u$ such that

$$|\mathrm{CG}(r, r-d, u)| = f_{d,u}(r) \quad \text{for all } r \geq 0.$$

If $u = 0$, then $f_{d,0}$ has degree $2d$, and the leading coefficient of $f_{d,0}$ is $1/(2^d \cdot d!)$.

To see that Theorem 3.8.2 implies Theorem 3.8.1, notice that

$$|\mathrm{CG}(r, r-d)| = \sum_{u=0}^{\infty} |\mathrm{CG}(r, r-d, u)| = \sum_{u=0}^{2d} |\mathrm{CG}(r, r-d, u)| = \sum_{u=0}^{2d} f_{d,u}(r),$$

where the middle equality follows from Theorem 3.7.1.

It remains to prove Theorem 3.8.2. The crucial observation here is that after r becomes large enough, no ‘new’ contracted graphs appear, and the only contracted graphs that do appear come from lower values of r. This is implied by the following lemmas.

Lemma 3.8.3. Let $\Gamma \in \mathrm{CG}(r, r-d, u)$ be a contracted graph. The number of marked vertices that have positive degree is at most $2d-u$. In particular, if $r > 2d-u$, there are marked vertices in Γ of degree 0.

Proof. Let $\Gamma \in \mathrm{CG}(r, r-d, u)$. Denote by r_+ the number of marked vertices with positive degree. As Γ is contracted, it follows that each unmarked vertex has degree at least 3. We obtain:

$$2e = \sum_{v \in \Gamma} \deg(v) \geq r_+ + 3u.$$

The characteristic of Γ equals $r-d$, so we have:

$$r - d = \chi = r + u - e.$$

Substituting $e = u + d$ into the above inequality yields $r_+ \leq 2d - u$. If $r > 2d - u$ we must conclude that $r_+ < r$, so Γ has a marked vertex of degree 0. □

Lemma 3.8.4. Let $0 \leq s \leq r$ be integers, and let $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ be an injection. Set $C = \{1, \ldots, r\} \setminus \mathrm{Im}(\phi)$. Recall that the pushforward functor induces a map

$$\phi_* : \mathrm{G}(s) \to \mathrm{G}(r).$$

This map is injective, and its image consists of those classes of r-marked graphs whose ith marked vertex has degree 0 for all $i \in C$.

Proof. Let $\Gamma = (V, E, m)$ be an s-marked graph, and let $\Gamma' = \phi_*\Gamma = (V', E', m')$. The following diagram is a pushout diagram

$$\begin{tikzcd}
\{1, \ldots, s\} \arrow[r, "m"] \arrow[d, "\phi"] & V \arrow[d, "\phi_{\mathrm{v}}"] \\
\{1, \ldots, r\} \arrow[r, "m'"'] \arrow[ur, phantom, "\lrcorner", very near end] & V'.
\end{tikzcd}$$

As m and ϕ are injective, it follows that:

$$\mathrm{Im}(\phi_{\mathrm{v}}) \cap \mathrm{Im}(m') = \mathrm{Im}(\phi_{\mathrm{v}} \circ m) = \mathrm{Im}(m' \circ \phi).$$

If $v' = m'(i)$ is a marked vertex of Γ' of positive degree, then v' must lie in the image of ϕ_{v}, as the endpoints of every edge in Γ' lie in the image of ϕ_{v}, by construction. As $v' = m'(i) \in \mathrm{Im}(\phi_{\mathrm{v}}) \cap \mathrm{Im}(m')$, we see that $i \in \mathrm{Im}(\phi)$, so $i \notin C$. In other words: for each $i \in C$ the marked vertex $m'(i)$ has degree 0.

Assume, now, that $\Gamma' = (V', E', m')$ is an r-marked graph such that the vertex $m'(i)$ has degree 0 for all $i \in C$. We construct an s-marked graph $\Gamma = (V, E, m)$ as follows: we let $V = V' \setminus m'(C)$, $E' = E$, and $m = m' \circ \phi$. It follows that this construction is inverse to the construction done in the first paragraph of this proof. This observation hinges on the fact that the following diagram is a pushout diagram.

$$\begin{tikzcd}
\{1, \ldots, s\} \arrow[r, "m' \circ \phi"] \arrow[d, "\phi"] \arrow[dr, phantom, "\lrcorner", very near end] & V' \setminus m(C) \arrow[d, hook] \\
\{1, \ldots, r\} \arrow[r, "m'"'] & V'.
\end{tikzcd}$$

It follows that ϕ_* induces a bijection from classes of s-marked graphs to classes of r-marked graphs whose ith marked vertex has degree 0 for all $i \in C$. ☐

Lemma 3.8.5. Let $0 \leq s \leq r$ be integers, and let $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ be an injection. The pushforward functor ϕ_* induces an injective map

$$\phi_* : \mathrm{CG}(s) \to \mathrm{CG}(r),$$

and hence injective maps for $d \geq 0$

$$\phi_* : \mathrm{CG}(s, s-d) \to \mathrm{CG}(r, r-d),$$

and for $d, u \geq 0$

$$\phi_* : \mathrm{CG}(s, s-d, u) \to \mathrm{CG}(r, r-d, u),$$

The images of these maps consist of the classes of those graphs whose ith marked vertex has degree 0 for all $i \in \{1, \ldots, r\} \setminus \mathrm{Im}(\phi)$.

Proof. The constructions made in the proof of Lemma 3.8.4 (deleting degree 0 marked vertices, taking pushforwards) do not affect unmarked vertices. Therefore it follows that an s-marked graph Γ is contracted if and only if $\phi_*\Gamma$ is contracted. As we observed in Section 3.3, the pushforward operator increases the characteristic of a graph by $r - s$, so ϕ_* maps graphs of characteristic $s - d$ to graphs of characteristic $r - d$. The number of unmarked vertices remains the same. ☐

We are almost ready to prove Theorem 3.8.2. In the proof of this theorem we will use a combinatorial argument to show that $|\mathrm{CG}(r, r-d, u)|$ can be expressed as a polynomial in r of degree at most $2d - u$. More precisely, we will show that $|\mathrm{CG}(r, r-d, u)|$ is fully determined by the values it takes for $0 \leq r \leq 2d - u$, and given by a recurrence relation. We will then apply the following recurrence relation for polynomials to see that $|\mathrm{CG}(r, r-d, u)|$ can be expressed as a polynomial in r.

Lemma 3.8.6. Let R be a commutative ring, let $r \geq 0$, and let $f \in R[x]$ a polynomial of degree less than r. Then

$$\sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x+k) = 0.$$

Proof. The forward difference operator $\Delta : R[x] \to R[x]$ maps a polynomial f to $f(x+1) - f(x)$. This map is R-linear, and maps polynomials of degree $\leq d$ to polynomials of degree $\leq d-1$. It follows that Δ^r annihilates all polynomials of degree less than r.

Moreover, one can prove using an inductive argument the identity

$$\Delta^r f = \sum_{k=0}^{r} (-1)^{r-k} \binom{r}{k} f(x+k),$$

for all $f \in R[x]$ and all $r \geq 0$. If $\deg f < r$ the desired identity follows. ☐

Proof of Theorem 3.8.2. If $u > 2d$, then there are no contracted r-marked graphs of characteristic $r-d$; this follows from Theorem 3.7.1.

Assume, from now on, that $u \leq 2d$. For each subset $A \subseteq \{1, \dots, r\}$ of cardinality k, we define a subset

$$S_A \subseteq \mathrm{CG}(r, r-d, u)$$

as follows: we let $\phi_A : \{1, \dots, r-k\} \to \{1, \dots, r\}$ denote the increasing map with image $\{1, \dots, r\} \setminus A$. Then S_A is the image of the (injective!) pushforward map

$$\phi_{A,*} : \mathrm{CG}(r-k, r-k-d, u) \to \mathrm{CG}(r, r-d, u).$$

It follows from Lemma 3.8.5 that a graph $\Gamma \in \mathrm{CG}(r, r-d, u)$ lies in S_A if and only if for all $i \in A$ the vertex $m(i)$ has degree 0. We therefore have, for subsets $A_1, \dots, A_m \subseteq \{1, \dots, r\}$:

$$S_{A_1} \cap \cdots \cap S_{A_m} = S_{A_1 \cup \cdots \cup A_m}.$$

If $r > 2d - u$, it follows from Lemma 3.8.3 that at least one of the marked vertices of each graph in $\mathrm{CG}(r, r-d, u)$ has degree 0. We can therefore write $\mathrm{CG}(r, r-d, u)$ as:

$$\mathrm{CG}(r, r-d, u) = \bigcup_{i=1}^{r} S_{\{i\}}.$$

The inclusion–exclusion principle then gives, for all $r > 2d - u$:

$$\begin{aligned}
|\mathrm{CG}(r, r-d, u)| &= \left| \bigcup_{i=1}^{r} S_{\{i\}} \right| \\
&= \sum_{k=1}^{r} (-1)^{k+1} \left(\sum_{\substack{A \subseteq \{1,\dots,r\} \\ |A| = k}} |S_A| \right) \\
&= \sum_{k=1}^{r} (-1)^{k+1} \binom{r}{k} |\mathrm{CG}(r-k, r-k-d, u)|.
\end{aligned}$$

So, if we let $g : \mathbb{Z}_{\geq 0} \to \mathbb{Z}_{\geq 0}$ denote the function

$$g(r) = |\mathrm{CG}(r, r-d, u)|,$$

we see that $g(r)$ is determined by its values in $0, \dots, 2d - u$ and the recursive formula

$$g(r) = \sum_{k=1}^{r} (-1)^{k+1} \binom{r}{k} g(r-k).$$

Let $f_{d,u}$ be the unique polynomial in $\mathbb{Q}[x]$ of degree at most $2d - u$ that satisfies $f_{d,u}(r) = g(r)$ for all $0 \leq r \leq 2d - u$. Using Lemma 3.8.6 one can show that $f_{d,u}$ satisfies the same recurrence relation as g does, and hence we conclude that $f_{d,u}(r) = g(r)$ for all $r \geq 0$.

Assume now that $u = 0$. Then $f_{d,0}(r)$ counts the number of contracted r-marked graphs with no unmarked vertices of characteristic $r - d$ (so the number of edges equals d). Every graph with no unmarked vertices is contracted, so $f_{d,0}(r)$ simply counts the number of ways we can put d edges in an r-marked graph with no unmarked vertices. There are up to ordering $\frac{1}{2}r(r+1)$ pairs of not necessarily distinct vertices in such a graph. Therefore, $f_{d,0}(r)$ equals the number of multisets of cardinality d with elements taken from a set of cardinality $\frac{1}{2}r(r+1)$. It follows that

$$f_{d,0}(r) = \left(\!\!\binom{\frac{1}{2}r(r+1)}{d}\!\!\right) = \binom{\frac{1}{2}r(r+1) + d - 1}{d}.$$

After expanding the binomial coefficient, we see that $f_{d,0}(r)$ is a degree $2d$ polynomial whose leading coefficient equals $1/(2^d \cdot d!)$. ☐

Remark 3.8.7. The polynomials $f_{d,u} \in \mathbb{Q}[x]$ are integer-valued. The coefficients of $f_{d,u}$, however, are not. By the theory of integer-valued polynomials (see, for instance, [CC16]), we can write

$$f_{d,u} = \sum_{k=0}^{2d-u} c_k \binom{x}{k},$$

where $c_0, \dots, c_{2d-u}$ are *integers*, defined recursively by the following formula:

$$c_k = f_{d,u}(k) - \sum_{j=0}^{k-1} c_j \binom{k}{i}.$$

3.9 Computing closed formulas for $|\mathrm{CG}(r, r-d)|$

In the previous section we proved that given nonnegative integers d, u there is a polynomial $f_{d,u}$ of degree at most $2d - u$ such that for all $r \geq 0$ one has

$$|\mathrm{CG}(r, r-d, u)| = f_{d,u}(r).$$

By Theorem 3.7.1 we then find for all $d, r \geq 0$:

$$|\mathrm{CG}(r, r-d)| = f_d(r) := \sum_{u=0}^{2d} f_{d,u}(r).$$

In this section we will use Algorithm 3.7.3 to compute the polynomial f_d for low values of d. An implementation of this algorithm in Python 3 (along with numerous optimizations to the 'naive' Algorithm 3.7.3) can be found in [vdLug21].

Using the algorithm, we can compute the polynomials $f_{d,u}$ and f_d for all $d \leq 4$ and $u \geq 0$.

- $d < 0$: By Theorem 3.8.2 we have $f_{d,u} = 0$ for all $u \geq 0$, and hence $f_d = 0$.
- $d = 0$: Theorem 3.8.2 implies that $f_{0,u} = 0$ for all $u > 0$. We have $f_{0,0} = 1$ since there is a unique r-marked graph with no unmarked vertices and no edges, and this graph is automatically contracted. We obtain $f_0 = 1$.
- $d = 1$: Using our algorithm, we find the following values for $f_{1,u}(r)$ for $r + u \leq 2$:

r	$f_{1,0}(r)$	$f_{1,1}(r)$	$f_{1,2}(r)$
0	0	1	1
1	1	1	
2	3		

Lagrange interpolation gives us the following expressions for the polynomials $f_{1,u}$:

$$\begin{aligned} f_{1,0} &= \tfrac{1}{2}r^2 + \tfrac{1}{2}r \\ f_{1,1} &= 1 \\ f_{1,2} &= 1 \end{aligned}$$

By summing these polynomials, we find

$$f_1 = \tfrac{1}{2}r^2 + \tfrac{1}{2}r + 2.$$

Note that this agrees with the formula for $\mathrm{CG}(r, r-1)$ we computed by hand in Example 3.7.2.

- $d = 2$: The algorithm produces the following values for $f_{2,u}(r)$ for $r + u \leq 4$:

r	$f_{2,0}(r)$	$f_{2,1}(r)$	$f_{2,2}(r)$	$f_{2,3}(r)$	$f_{2,4}(r)$
0	0	1	4	3	3
1	1	5	8	4	
2	6	13	14		
3	21	26			
4	55				

By interpolation, we find the polynomial equations

$$\begin{aligned} f_{2,0} &= \tfrac{1}{8}r^4 + \tfrac{1}{4}r^3 + \tfrac{3}{8}r^2 + \tfrac{1}{4}r \\ f_{2,1} &= \tfrac{1}{6}r^3 + \tfrac{3}{2}r^2 + \tfrac{7}{3}r + 1 \\ f_{2,2} &= r^2 + 3r + 4 \\ f_{2,3} &= r + 3 \\ f_{2,4} &= 3. \end{aligned}$$

Hence we obtain

$$f_2 = \tfrac{1}{8}r^4 + \tfrac{5}{12}r^3 + \tfrac{23}{8}r^2 + \tfrac{79}{12}r + 11.$$

- $d = 3$: We find the following values for $f_{3,u}(r)$ for $r + u \leq 6$:

r	$f_{3,0}(r)$	$f_{3,1}(r)$	$f_{3,2}(r)$	$f_{3,3}(r)$	$f_{3,4}(r)$	$f_{3,5}(r)$	$f_{3,6}(r)$
0	0	1	7	18	23	15	9
1	1	10	33	49	44	20	
2	10	51	104	106	73		
3	56	176	257	197			
4	220	475	541				
5	680	1086					
6	1771						

Interpolation of the found data yields the following polynomial expressions.

$$\begin{aligned} f_{3,0} &= \tfrac{1}{48}r^6 + \tfrac{1}{16}r^5 + \tfrac{3}{16}r^4 + \tfrac{13}{48}r^3 + \tfrac{7}{24}r^2 + \tfrac{1}{6}r \\ f_{3,1} &= \tfrac{1}{12}r^5 + \tfrac{3}{4}r^4 + \tfrac{25}{12}r^3 + \tfrac{13}{4}r^2 + \tfrac{17}{6}r + 1 \\ f_{3,2} &= \tfrac{1}{2}r^4 + \tfrac{19}{6}r^3 + \tfrac{19}{2}r^2 + \tfrac{77}{6}r + 7 \\ f_{3,3} &= \tfrac{4}{3}r^3 + 9r^2 + \tfrac{62}{3}r + 18 \\ f_{3,4} &= 4r^2 + 17r + 23 \\ f_{3,5} &= 5r + 15 \\ f_{3,6} &= 9 \end{aligned}$$

We therefore find that the number of contracted r-marked graphs of characteristic $r - 3$ is equal to:

$$f_3 = \tfrac{1}{48}r^6 + \tfrac{7}{48}r^5 + \tfrac{23}{16}r^4 + \tfrac{329}{48}r^3 + \tfrac{625}{24}r^2 + \tfrac{117}{2}r + 73$$

- $d = 4$: After a while the algorithm outputs the following values for $f_{4,u}(r)$ for $u + r \leq 8$.

r	$f_{4,0}(r)$	$f_{4,1}(r)$	$f_{4,2}(r)$	$f_{4,3}(r)$	$f_{4,4}(r)$	$f_{4,5}(r)$	$f_{4,6}(r)$	$f_{4,7}(r)$	$f_{4,8}(r)$
0	0	1	11	47	123	172	160	79	32
1	1	16	93	257	425	423	282	105	
2	15	136	496	948	1131	854	443		
3	126	742	1897	2707	2513	1515			
4	715	2971	5756	6485	4916				
5	3060	9542	14 786	13 687					
6	10 626	26 047	33 538						
7	31 465	62 812							
8	82 251								

We therefore obtain the following polynomial expressions for $f_{4,u}(r)$:

$$\begin{aligned}
f_{4,0} &= \tfrac{1}{384}r^8 + \tfrac{1}{96}r^7 + \tfrac{3}{64}r^6 + \tfrac{5}{48}r^5 + \tfrac{27}{128}r^4 + \tfrac{25}{96}r^3 + \tfrac{23}{96}r^2 + \tfrac{1}{8}r \\
f_{4,1} &= \tfrac{1}{48}r^7 + \tfrac{5}{24}r^6 + \tfrac{101}{120}r^5 + \tfrac{9}{4}r^4 + \tfrac{63}{16}r^3 + \tfrac{109}{24}r^2 + \tfrac{16}{5}r + 1 \\
f_{4,2} &= \tfrac{23}{144}r^6 + \tfrac{73}{48}r^5 + \tfrac{1013}{144}r^4 + \tfrac{875}{48}r^3 + \tfrac{1037}{36}r^2 + \tfrac{105}{4}r + 11 \\
f_{4,3} &= \tfrac{3}{4}r^5 + \tfrac{23}{3}r^4 + \tfrac{397}{12}r^3 + \tfrac{229}{3}r^2 + \tfrac{553}{6}r + 47 \\
f_{4,4} &= \tfrac{73}{24}r^4 + \tfrac{325}{12}r^3 + \tfrac{2387}{24}r^2 + \tfrac{2069}{12}r + 123 \\
f_{4,5} &= \tfrac{25}{3}r^3 + 65r^2 + \tfrac{533}{3}r + 172 \\
f_{4,6} &= \tfrac{39}{2}r^2 + \tfrac{205}{2}r + 160 \\
f_{4,7} &= 26r + 79 \\
f_{4,8} &= 32
\end{aligned}$$

By summing these polynomials, we obtain the polynomial f_4 that counts the number of contracted r-marked graphs of characteristic $r - 4$.

$$f_4 = \tfrac{1}{384}r^8 + \tfrac{1}{32}r^7 + \tfrac{239}{576}r^6 + \tfrac{193}{60}r^5 + \tfrac{23275}{1152}r^4 + \tfrac{8729}{96}r^3 + \tfrac{84637}{288}r^2 + \tfrac{24013}{40}r + 625$$

While our algorithm can quickly compute all values for $f_{d,u}(r)$ with $u + r \leq 2d$ for all $d \leq 3$, it takes a very long time in the case $d = 4$ on the same server. We observe the following runtimes:

d	Runtime (s)
1	4.3×10^{-4}
2	2.3×10^{-2}
3	3.6
4	8.2×10^{3}

It seems unlikely that f_5 can be computed in reasonable time without significant improvements to either the algorithm or the hardware.

Chapter 4

Tautological differential forms on moduli of curves

In this section we will establish a theory of tautological differential forms on families of curves, that is meant to give an analytic analogue to the theory of tautological rings and tautological cohomology. We first discuss a suitable definition for the rings of tautological forms. This definition, however, introduces exact tautological forms that cannot be detected by cohomology; it follows that the rings of tautological forms are 'bigger' than the rings of tautological classes. Next, we will describe a combinatorial framework, using marked graphs, that allows us to generate tautological forms, and prove that in fact all tautological forms can be constructed in this way, thereby showing that the rings of tautological forms are not 'too big'. Finally, we describe a method for generating relations in the rings of tautological forms and fully compute the degree 2 parts of these rings.

4.1 Tautological morphisms and submersions

Fix an integer $g \geq 2$. In Chapter 2 we have defined the universal family $p : \mathcal{C}_g \to \mathcal{M}_g$ of genus g curves. Although $\mathcal{M}_g$ and $\mathcal{C}_g$ are not complex manifolds but merely differentiable stacks, we will often treat these spaces as if they were honest manifolds. The reader should understand that statements about this universal family of genus g curves can in that case be interpreted as statements that hold universally among all families of genus g curves. In Chapter 2 we have clarified this correspondence between statements for the universal family and universal statements for families.

Let us briefly recall the tautological morphisms we constructed in Chapter 2. Let $f : \mathcal{C} \to S$ be a family of genus g curves. Recall that to each pair of integers $r, s \geq 0$ and each map of sets $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ we have associated a morphism

$$f^{\phi} : \mathcal{C}^r \to \mathcal{C}^s : (x_1, \ldots, x_r) \mapsto (x_{\phi(1)}, \ldots, x_{\phi(s)}),$$

where $\mathcal{C}^r$ and $\mathcal{C}^s$ denote the r-fold and s-fold fiber products of $\mathcal{C}$ over S. This morphism is a submersion if and only if ϕ is injective. Universally, we obtain a morphism

$$f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$$

and morphisms of this form are called tautological morphisms. The tautological morphism f^ϕ is a submersion if and only if ϕ is injective.

The following examples list some tautological morphisms that we will be using often.

Example 4.1.1. The tautological morphism associated to the unique map $\{1,2\} \to \{1\}$ is the diagonal morphism $\Delta : \mathcal{C}_g \to \mathcal{C}_g^2$.

Example 4.1.2. The tautological submersion associated to the unique map $\emptyset \to \{1,\dots,r\}$ is the projection morphism $\mathcal{C}_g^r \to \mathcal{M}_g$.

Example 4.1.3. If $1 \le i \le r$ is an integer, the map $\{1\} \to \{1,\dots,r\}$ given by $1 \mapsto i$ induces the map $\mathcal{C}_g^r \to \mathcal{C}_g$ that projects onto the ith coordinate. We denote this map by p_i.
More generally, if $1 \le i_1,\dots,i_s \le r$ are integers, we denote by

$$p_{i_1,\dots,i_s} : \mathcal{C}_g^r \to \mathcal{C}_g^s$$

the tautological morphism associated to $\phi : \{1,\dots,s\} \to \{1,\dots,r\} : k \mapsto i_k$.

Example 4.1.4. Let $1 \le i_1 < \cdots < i_s \le r$ be integers. Consider the unique increasing map $\phi : \{1,\dots,r-s\} \to \{1,\dots,r\}$ whose image is $\{1,\dots,r\}\setminus\{i_1,\dots,i_s\}$. Denote by

$$p_{(i_1,\dots,i_s)} : \mathcal{C}_g^r \to \mathcal{C}_g^{r-s}$$

the tautological morphism associated to ϕ (notice the parentheses!). Then $p_{(i_1,\dots,i_s)}$ is the tautological submersion that 'forgets the coordinates $i_1,\dots,i_s$'. For instance, the map $p_{(2)} : \mathcal{C}_g^2 \to \mathcal{C}_g$ equals the map $p_1 : \mathcal{C}_g^2 \to \mathcal{C}_g$.

Consider a commutative diagram of sets, together with the associated diagram of moduli stacks.

$$\begin{tikzcd}
\{1,\dots,u\} & \{1,\dots,s\} \arrow[l, "\eta"'] \\
\{1,\dots,t\} \arrow[u, "\chi"'] & \{1,\dots,r\} \arrow[l, "\psi"'] \arrow[u, "\phi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_g^u \arrow[r, "f^\eta"] \arrow[d, "f^\chi"] & \mathcal{C}_g^s \arrow[d, "f^\phi"] \\
\mathcal{C}_g^t \arrow[r, "f^\psi"] & \mathcal{C}_g^r
\end{tikzcd}$$

As we have seen in Section 2.4, the diagram of moduli stacks is cartesian if and only if the diagram of sets is a pushout diagram. We will be using such cartesian diagrams often.

4.2 Tautological classes

This section serves as a short introduction to tautological rings of moduli spaces of curves. We will recall the definition of the tautological ring $R^*(\mathcal{C}_g^r)$ of $\mathcal{C}_g^r$, which is a subring of the Chow ring $\mathrm{CH}^*(\mathcal{C}_g^r)$ of $\mathcal{C}_g^r$ with rational coefficients.

Let $g \geq 2$ be an integer, and consider the universal family $p : \mathcal{C}_g \to \mathcal{M}_g$ of genus g curves. We let $\omega_{\mathcal{C}_g/\mathcal{M}_g}$ denote the relative cotangent bundle, and $K \in \mathrm{CH}^1(\mathcal{C}_g)$ its first Chern class in the Chow ring with rational coefficients. For $d \geq 0$ we define the dth *Mumford–Morita–Miller class* κ_d by

$$\kappa_d = p_* K^{d+1} \in \mathrm{CH}^d(\mathcal{M}_g).$$

The *tautological ring* on $\mathcal{M}_g$, defined by Mumford [Mum83], is the sub-$\mathbb{Q}$-algebra $R^*(\mathcal{M}_g) \subseteq \mathrm{CH}^*(\mathcal{M}_g)$ generated by these κ-classes. Mumford proved that the tautological ring is generated by the tautological classes $\kappa_1, \ldots, \kappa_{g-2}$. He also proved that all Chern classes of the Hodge bundle $p_*\omega_{\mathcal{C}_g/\mathcal{M}_g}$ lie in the tautological ring.

The Chow ring and the tautological ring vanish in degrees higher than $\dim(\mathcal{M}_g) = 3g-3$. Looijenga [Loo95] proved the stronger statement that $R^*(\mathcal{M}_g)$ vanishes in degrees higher than $g-2$, and that $R^{g-2}(\mathcal{M}_g)$ is at most one-dimensional, spanned by the class κ_{g-2}. Faber [Fab97] then proved that κ_{g-2} is nonzero, so $R^{g-2}(\mathcal{M}_g)$ is one-dimensional. Faber also conjectured that the tautological ring is a *Gorenstein algebra.*

Conjecture 4.2.1 ([Fab99])**.** For any $g \geq 2$ the following holds.

1. $R^d(\mathcal{M}_g) = 0$ for $d > g-2$;
2. $R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$;
3. Multiplication in the Chow ring gives a perfect pairing

$$R^d(\mathcal{M}_g) \times R^{g-2-d}(\mathcal{M}_g) \to R^{g-2}(\mathcal{M}_g) \cong \mathbb{Q}$$

for all $0 \leq d \leq g-2$.

This conjecture has been verified by Faber [Fab13] for all $g \leq 23$, but not enough relations have been found in genus 24 to verify the conjecture there.

More generally, the *tautological ring* $R^*(\mathcal{C}_g^r)$ of $\mathcal{C}_g^r$ (introduced in [Loo95]) is defined to be the $\mathbb{Q}$-subalgebra of $\mathrm{CH}^*(\mathcal{C}_g^r)$ generated by the following classes:

- the classes κ_d (obtained from $\mathcal{M}_g$ by pullback);
- the classes $K_i = p_i^* K$ for $1 \leq i \leq r$;
- the diagonal classes $\Delta_{ij} = p_{ij}^*\Delta$, with $\Delta \subseteq \mathcal{C}_g^2$ the diagonal, for $1 \leq i < j \leq r$.

Note that the classes K_i can also be defined as follows: if $p_{(i)} : \mathcal{C}_g^r \to \mathcal{C}_g^{r-1}$ is the projection map that forgets the ith coordinate, then K_i is the first Chern class of the relative cotangent bundle of this projection.

In the ring $R^*(\mathcal{C}_g^r)$ we have the relations:

$$\begin{aligned} \Delta_{ij}\Delta_{jk} &= \Delta_{ij}\Delta_{ik} \\ \Delta_{ij}^2 &= -\,\Delta_{ij}K_i. \end{aligned}$$

where i, j, k are pairwise distinct. If $\Delta : \mathcal{C}_g \to \mathcal{C}_g^2$ denotes the diagonal map, we have:

$$\Delta^*\Delta = -K.$$

Looijenga proved in [Loo95] that $R^*(\mathcal{C}_g^r)$ vanishes in degree $d > g + r - 2$.

Using the above relations, one can deduce that the pullbacks along tautological morphisms of tautological classes are again tautological classes. Moreover, it is straightforward to verify using the above relations and the projection formula that the pushforward of every tautological class along every tautological morphism is a tautological class. In other words: the system of $\mathbb{Q}$-algebras $\{R^*(\mathcal{C}_g^r) : r \geq 0\}$ is closed under pushforward and pullback along tautological morphisms. If $\{S^*(\mathcal{C}_g^r) : r \geq 0\}$ is another system of $\mathbb{Q}$-subalgebras of the Chow rings that is closed under pushforwards and pullbacks along tautological morphisms, then

$$\begin{aligned} \Delta &= \Delta_*(1) \in S^*(\mathcal{C}_g^2) \\ K &= -\Delta^*\Delta \in S^*(\mathcal{C}_g) \\ \kappa_d &= p_*K^{d+1} \in S^*(\mathcal{M}_g). \end{aligned}$$

It follows that the classes κ_d, K_i and Δ_{ij} lie in $S^*(\mathcal{C}_g^r)$, and therefore $R^*(\mathcal{C}_g^r) \subseteq S^*(\mathcal{C}_g^r)$. We obtain the following.

Proposition 4.2.2. The system of $\mathbb{Q}$-subalgebras $R^*(\mathcal{C}_g^r) \subseteq \mathrm{CH}^*(\mathcal{C}_g^r)$ (with $r \geq 0$) is the smallest system of $\mathbb{Q}$-subalgebras that is closed under pullbacks and pushforwards along tautological morphisms. ☐

In fact, we can slightly rephrase this proposition to the following. It will be this formulation that allows us to translate the language of tautological classes to a language of tautological differential forms.

Proposition 4.2.3. The system of $\mathbb{Q}$-subalgebras $R^*(\mathcal{C}_g^r) \subseteq \mathrm{CH}^*(\mathcal{C}_g^r)$ (with $r \geq 0$) satisfies:

1. $\Delta \in R^*(\mathcal{C}_g^2)$;
2. the system is closed under pullbacks along tautological morphisms;
3. the system is closed under pushforwards along tautological *submersions*;
4. the system is the smallest system that satisfies 1–3. ☐

Analogously, the *tautological cohomology ring* $RH^*(\mathcal{C}_g^r)$ is a subring of the cohomology ring of $\mathcal{C}_g^r$ with rational coefficients. It is defined as the image of the canonical map

$$R^*(\mathcal{C}_g^r) \to H^{2*}(\mathcal{C}_g^r, \mathbb{Q}).$$

Notice that the grading on cohomology is twice the grading in the Chow ring, and that the tautological cohomology ring does not contain any odd-degree cohomology classes. So far, it seems to be unknown whether the canonical map from tautological Chow classes to tautological cohomology classes is an isomorphism.

We will define a third tautological ring of a more analytical nature, the ring of tautological differential forms, which is a subring of the ring of real differential forms on $\mathcal{C}_g^r$. These forms will be closed differential forms and we can take their cohomology classes in $H^*(\mathcal{C}_g^r, \mathbb{R})$. When we compare rings of tautological differential forms with tautological cohomology rings, we should consider cohomology with real coefficients.

4.3 Rings of tautological differential forms

Fix an integer $g \geq 2$. In Section 4.2 we have seen that there are multiple equivalent ways to define the rings of tautological Chow or cohomology classes on the moduli stacks $\mathcal{C}_g^r$. A priori, these rings are defined to be the sub-$\mathbb{Q}$-algebras of the Chow or cohomology rings that are generated by the classes Δ_{ij}, K_i and κ_d, and Propositions 4.2.2 and 4.2.3 yield two more equivalent definitions. In this section, we will attempt to translate these definitions to an analytical setting. Rather than Chern or cohomology classes, we will consider differential forms. Of the three equivalent definitions for the ring of tautological forms given in Section 4.2, the one given by Proposition 4.2.3 can be translated directly to the analytical setting, and we will be using this translation to define rings of tautological differential forms on the moduli stacks $\mathcal{C}_g^r$.

Let us endow the line bundle $O(\Delta)$ on $\mathcal{C}_g^2$ with its canonical metric (see Section 1.4), and take the first Chern form of the resulting hermitian bundle to obtain a closed real 2-form

$$h = c_1(O(\Delta))$$

on $\mathcal{C}_g^2$ that represents the diagonal. Let $\omega = \omega_{\mathcal{C}_g/\mathcal{M}_g}$ be the relative cotangent bundle of the universal family of genus g curves $\mathcal{C}_g \to \mathcal{M}_g$, endowed with its canonical metric; recall from Section 1.4 that we have a canonical isometry

$$\omega^{\otimes -1} \simeq \Delta^* O(\Delta)$$

of hermitian vector bundles on $\mathcal{C}_g$. We therefore have

$$c_1(\omega) = -\Delta^* c_1(O(\Delta)) = -e^A \in A^2(\mathcal{C}_g)$$

where e^A is defined to be the first Chern form of the relative tangent bundle

$$T_{\mathcal{C}_g/\mathcal{M}_g} \simeq \omega^{\otimes -1} \simeq \Delta^* O(\Delta)$$

with the metric induced by the canonical metric on ω.

Recall that in the Chow ring $\mathrm{CH}^*(\mathcal{M}_g)$ we have constructed the kappa-classes κ_d by pushing forward powers of the canonical class K on $\mathcal{C}_g$ along the universal family $\mathcal{C}_g \to \mathcal{M}_g$. Analogously we define forms $e_d^A \in A^{2d}(\mathcal{M}_g)$ for all $d \geq 0$ by

$$e_d^A := \int_{\mathcal{C}_g/\mathcal{M}_g} (e^A)^{d+1}.$$

Let us consider the sub-$\mathbb{R}$-algebras of $A^*(\mathcal{C}_g^r)$ generated by forms $p_{ij}^* h$, $p_i^* e^A$ and e_d^A. Certainly, we want to consider forms in these rings to be tautological. However, a problem arises: this system of rings is not closed under fiber integrals along projection maps. For instance, consider the differential form

$$\nu := \int_{\mathcal{C}_g^2/\mathcal{M}_g} h^3 \in A^2(\mathcal{M}_g).$$

We have [dJon16]:

$$\nu - e_1^A = \frac{\partial\bar{\partial}\varphi}{\pi\sqrt{-1}},$$

where $\varphi \in A^0(\mathcal{M}_g)$ is the Kawazumi-Zhang invariant, introduced by Kawazumi [Kaw08; Kaw09] and Zhang [Zha10] in different contexts. Later we will see that for $g \geq 3$ the forms ν and e_1^A are linearly independent (whereas for $g = 2$ there is a linear relation), and thus we find that ν is not in the subring of $A^*(\mathcal{M}_g)$ generated by the classes e_d^A for all $g \geq 3$.

A second problem is the fact that, in the context of differential forms, proper pushforwards or fiber integrals can only be taken along submersions. While the tautological class Δ on $\mathcal{C}_g^2$ can be obtained by taking the pushforward of 1 along the diagonal map $\mathcal{C}_g \to \mathcal{C}_g^2$, we can not obtain the corresponding form h in an analogous way.

The following definition, based on Proposition 4.2.3, solves both our problems.

Definition 4.3.1. The *rings of tautological forms* $\mathcal{R}^*(\mathcal{C}_g^r)$ $(r \geq 0)$ are the unique sub-$\mathbb{R}$-algebras $\mathcal{R}^*(\mathcal{C}_g^r) \subseteq A^*(\mathcal{C}_g^r)$ such that the following holds:

1. $h \in \mathcal{R}^*(\mathcal{C}_g^2)$;
2. If $f : \mathcal{C}_g^r \to \mathcal{C}_g^s$ is a tautological morphism, then $f^*(\mathcal{R}^*(\mathcal{C}_g^s)) \subseteq \mathcal{R}^*(\mathcal{C}_g^r)$;
3. If $f : \mathcal{C}_g^r \to \mathcal{C}_g^s$ is a tautological submersion, then $\int_f(\mathcal{R}^*(\mathcal{C}_g^r)) \subseteq \mathcal{R}^*(\mathcal{C}_g^s)$;
4. $\mathcal{R}^*(\mathcal{C}_g^r)$ are minimal: if $S^*(\mathcal{C}_g^r) \subseteq A^*(\mathcal{C}_g^r)$ $(r \geq 0)$ is another collection of sub-$\mathbb{R}$-algebras that satisfies 1–3, then $\mathcal{R}^*(\mathcal{C}_g^r) \subseteq S^*(\mathcal{C}_g^r)$ for all $r \geq 0$.

Elements of these rings are called *tautological (differential) forms.*

Notice that this definition implies that there are no tautological forms of odd degree. Indeed, taking pullbacks and fiber integrals of differential forms along morphisms of complex manifolds changes the degrees of these differential forms by an even number; see Proposition 1.3.19. It follows that removing the summands of odd degree from the rings

$$\mathcal{R}^*(\mathcal{C}_g^r) = \bigoplus_{d \geq 0} \mathcal{R}^d(\mathcal{C}_g^r)$$

still yields a system that satisfies properties 1–3, which is smaller than, and hence equal to, the system of tautological differential forms.

Definition 4.3.1 implies that $e^A = \Delta^* h$ is a tautological form on $\mathcal{C}_g$, and $e_d^A = \int_p (e^A)^{d+1}$ is a tautological form on $\mathcal{M}_g$ for all $d \geq 0$. It follows that passing to cohomology yields a surjective map

$$\mathcal{R}^*(\mathcal{C}_g^r) \to RH^*(\mathcal{C}_g^r) \otimes_{\mathbb{Q}} \mathbb{R}.$$

However, as opposed to the settings of Chow rings and cohomology rings, the tautological rings are not generated by pullbacks of classes h, e^A and e^A_d. For instance, the real 2-form

$$\frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}}$$

is a tautological form on $\mathcal{M}_g$ that is not in the subring generated by the e^A_d if $g \geq 3$. Such 'extra' tautological forms are introduced by the homogeneous ideal $I^*(\mathcal{C}_g^r)$ of exact tautological forms:

$$0 \to I^*(\mathcal{C}_g^r) \to \mathcal{R}^*(\mathcal{C}_g^r) \to RH^*(\mathcal{C}_g^r) \otimes_{\mathbb{Q}} \mathbb{R} \to 0.$$

By Looijenga's result [Loo95] we know that all tautological forms of degree $d > 2(g + r - 2)$ are exact.

Of particular interest is the degree 2 part

$$I^2(\mathcal{M}_g) \subseteq I^*(\mathcal{M}_g).$$

If $g \geq 3$ then exact forms in this space can be written in the form

$$\frac{\partial\overline{\partial}\alpha}{\pi\sqrt{-1}}$$

with α a real-valued smooth function on $\mathcal{M}_g$ defined uniquely up to an additive constant; see [Kaw09, Lemma 8.1]. For example, the Kawazumi–Zhang invariant φ arises from the exact tautological form $\nu - e^A_1$ in this way. One might wonder if it is possible to obtain more such invariants for genus g curves from exact tautological 2-forms on $\mathcal{M}_g$. As it will turn out, this is not the case. In Corollary 4.8.4 we will find that $I^2(\mathcal{M}_g)$ is spanned by

$$\frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}},$$

and that the Kawazumi–Zhang invariant is the only invariant, up to additive and multiplicative constants, that arises in this way.

Next, we will prove some elementary equalities of tautological differential forms, which we will use in the proof of Proposition 4.6.2.

Lemma 4.3.2. Let $p : \mathcal{C}_g \to \mathcal{M}_g$ be the universal family of genus g curves, and let e^A be the first Chern form of the relative tangent bundle $T_{\mathcal{C}_g/\mathcal{M}_g} \simeq \omega^{\otimes -1}$ with its canonical metric. Then

$$\int_p e^A = 2 - 2g \in A^0(\mathcal{M}_g).$$

Proof. Recall that the cotangent bundle of any genus g curve has degree $2g - 2$. Applying Lemma 1.4.10 therefore gives the desired result. ☐

Lemma 4.3.3. Consider the tautological submersion $p_1 : \mathcal{C}_g^2 \to \mathcal{C}_g$. Then

$$\int_{p_1} h = 1 \in A^0(\mathcal{C}_g).$$

Proof. This identity, too, follows immediately from Lemma 1.4.10. ☐

Lemma 4.3.4. Consider the tautological submersion $p_1 : \mathcal{C}_g^2 \to \mathcal{C}_g$. If L is a hermitian line bundle on $\mathcal{C}_g^2$ which is fiberwise admissible with respect to p_1, then

$$\int_{p_1} h \wedge c_1(L) = \Delta^* c_1(L) \in A^2(\mathcal{C}_g).$$

In particular, we have:

$$\int_{p_1} h^2 = e^A$$

and for $i = 1, 2$ we have

$$\int_{p_1} h \wedge p_i^* e^A = e^A.$$

Proof. From Proposition 1.4.13 we obtain

$$\int_{p_1} h \wedge c_1(L) = \int_{p_1} c_1(O(\Delta)) \wedge c_1(L) = c_1(\langle O(\Delta), L\rangle_{p_1}) = c_1(\Delta^* L) = \Delta^* c_1(L),$$

where the third equality follows from the fact that the canonical metric on $O(\Delta)$ has the useful property that the canonical isomorphism

$$\langle O(\Delta), L\rangle_{p_1} \xrightarrow{\sim} \Delta^* L$$

is an isometry; see Section 1.4. The other identities now follow from:

$$h = c_1(O(\Delta)), \quad \text{and} \quad p_i^* e^A = p_i^* c_1(\omega^{\otimes -1}) = c_1(p_i^* \omega^{\otimes -1}).$$ ☐

Lemma 4.3.5. Let $p_{12}, p_{13}, p_{23} : \mathcal{C}_g^3 \to \mathcal{C}_g^2$ be the three tautological submersions. Then

$$\int_{p_{12}} p_{13}^* h \wedge p_{23}^* h = h \in A^2(\mathcal{C}_g^2).$$

Proof. Let $\sigma_1, \sigma_2 : \mathcal{C}_g^2 \to \mathcal{C}_g^3$ be the two canonical sections of p_{12}, such that $p_3 \circ \sigma_i = p_i : \mathcal{C}_g^2 \to \mathcal{C}_g$ for $i = 1, 2$. Notice that $p_{13} \circ \sigma_2 : \mathcal{C}_g^2 \to \mathcal{C}_g^2$ is the identity. Endow the induced line bundles $O(\sigma_1), O(\sigma_2)$ on $\mathcal{C}_g^3$ with their canonical metrics. We use

Proposition 1.4.13 to obtain

$$\begin{aligned}\int_{p_{12}} p_{13}^* h \wedge p_{23}^* h &= \int_{p_{12}} p_{13}^* c_1(O(\Delta)) \wedge p_{23}^* c_1(O(\Delta)) \\ &= \int_{p_{12}} c_1(O(\sigma_1)) \wedge c_1(O(\sigma_2)) \\ &= c_1(\langle O(\sigma_1), O(\sigma_2)\rangle_{p_1}) \\ &= \sigma_2^* c_1(O(\sigma_1)) \\ &= \sigma_2^* p_{13}^* c_1(O(\Delta)) \\ &= c_1(O(\Delta)) \\ &= h.\end{aligned}$$

□

Recall from Section 1.4 that for each family $f : \mathcal{C} \to S$ of genus g curves with Jacobian family $\mathcal{J} \to S$ we have canonical morphisms $\kappa : \mathcal{C} \to \mathcal{J}$ and $\delta : \mathcal{C}^2 \to \mathcal{J}$. The morphism κ takes a point x in a fiber $\mathcal{C}_s$ and maps it to the class of the degree 0 line bundle $O((2g-2)x) \otimes \omega_{\mathcal{C}_s}^{\otimes -1}$ in $\mathcal{J}_s = \mathrm{Jac}(\mathcal{C}_s)$. The morphism δ is the Abel–Jacobi morphism: it maps a pair $(x, y) \in \mathcal{C}_s^2$ to the class of the line bundle $O(y - x)$ in $\mathcal{J}_s = \mathrm{Jac}(\mathcal{C}_s)$.

Universally we obtain morphisms $\kappa : \mathcal{C}_g \to \mathcal{J}_g$ and $\delta : \mathcal{C}_g^2 \to \mathcal{J}_g$. Recall from Section 2.7 that on the universal Jacobian $\mathcal{J}_g$ we have constructed a canonical hermitian line bundle $\mathcal{B}$. We denote by $2\omega_0$ the first Chern form of $\mathcal{B}$. As the form $2\omega_0$ and the morphisms κ, δ are completely canonical, it makes sense to expect that the forms $2\kappa^*\omega_0$ and $2\delta^*\omega_0$ are tautological. Indeed, this is the case, as the following proposition shows.

Proposition 4.3.6. The forms $\kappa^*\omega_0 \in A^2(\mathcal{C}_g)$ and $\delta^*\omega_0 \in A^2(\mathcal{C}_g^2)$ are tautological. More precisely, we have the following identities of 2-forms:

$$\begin{aligned}-2\kappa^*\omega_0 &= 2g(2g-2)e^A + p^* e_1^A \\ -2\delta^*\omega_0 &= p_1^* e^A + p_2^* e^A - 2h.\end{aligned}$$

Note that these identities match identities (K1) and (K3) in [dJon16, Theorem 1.4].

Proof. Denote by $p : \mathcal{C}_g \to \mathcal{M}_g$ the universal family of genus g curves. Recall from Proposition 1.4.15 that we have canonical isometries

$$\begin{aligned}\kappa^*\mathcal{B}^{\otimes -1} &\simeq \omega^{-2g(2g-2)} \otimes p^*\langle \omega, \omega\rangle_p \\ \delta^*\mathcal{B}^{\otimes -1} &\simeq p_1^*\omega^{\otimes -1} \otimes p_2^*\omega^{\otimes -1} \otimes O(\Delta)^{\otimes -2}\end{aligned}$$

Taking first Chern classes and applying Proposition 1.4.13 then yields the desired result. □

One could argue, in fact, that the 2-form $2\delta^*\omega_0$ is the 'prototypical' tautological form on $\mathcal{C}_g^2$, more so than h, and replace h by $2\delta^*\omega_0$ in Definition 4.3.1. We claim that this does not affect the resulting system of tautological rings. Indeed, Proposition 4.3.6 states that $2\delta^*\omega_0$ is tautological. Conversely, it is possible to obtain h from $2\delta^*\omega_0$ by using pullbacks and fiber integrals as follows. Squaring $-2\delta^*\omega_0 = p_1^* e^A + p_2^* e^A - 2h$ and integrating the result along the fibers of $p_1 : \mathcal{C}_g^2 \to \mathcal{C}_g$ yields:

$$\begin{aligned}\int_{p_1} (-2\delta^*\omega_0)^2 = \int_{p_1} c_1(\delta^*\mathcal{B}^{\otimes -1})^2 &= c_1(\langle \mathcal{B}^{\otimes -1}, \mathcal{B}^{\otimes -1}\rangle_{p_1}) \\ &= c_1(\omega^{\otimes 4g} \otimes p^*\langle \omega, \omega\rangle_p) = -4ge^A + p^*e_1^A,\end{aligned}$$

where the second and third equalities follow from Propositions 1.4.13 and 1.4.15, respectively. Squaring the resulting form and integrating it along the fibers of $p : \mathcal{C}_g \to \mathcal{M}_g$ then gives:

$$\int_p (-4ge^A + p^*e_1^A)^2 = \int_p (16g^2(e^A)^2 - 8ge^A \wedge p^*e_1^A + p^*(e_1^A)^2)$$

We have:

$$\int_p 16g^2(e^A)^2 = 16g^2 e_1^A,$$

and applying the projection formula and Lemma 4.3.2 yields

$$\int_p -8ge^A \wedge p^*e_1^A = -8ge_1^A \wedge \int_p e^A = -8g(2-2g)e_1^A.$$

Another application of the projection formula gives

$$\int_p p^*(e_1^A)^2 = (e_1^A)^2 \int_p 1 = 0.$$

We conclude:

$$\int_p (-4ge^A + p^*e_1^A)^2 = 16g(2g-1)e_1^A.$$

We thus find that we can obtain e_1^A, e^A, and finally h from $2\delta^*\omega_0$ by taking fiber integrals and pullbacks.

4.4 Tautological forms associated to marked graphs

Now that we have defined the rings of tautological forms, we need a method to generate lots of tautological forms in order to be able to study relations of these forms. We can start with some ‘basic’ tautological forms like h and e^A and take pullbacks, fiber integrals, and wedge products in order to generate more tautological forms. The theory of marked graphs gives us a combinatorial framework for generating such forms, and it will turn out that this framework is able to give us all tautological forms.

In this section, we fix an integer $g \geq 2$, and we will describe an operation that takes an r-marked graph and outputs a tautological form on $\mathcal{C}_g^r$.

Let $\Gamma = (V, E, m)$ be an r-marked graph, and let u be the number of unmarked vertices of Γ. Choose a bijective extension

$$\bar{m} : \{1, \ldots, r+u\} \xrightarrow{\sim} V$$

of the marking $m : \{1, \ldots, r\} \to V$. We will define a differential form μ_Γ on $\mathcal{C}_g^{r+u}$ that will depend on the choice of this extension $\bar{m}$.

First, we associate to every edge $e \in E$ a 2-form h_e on $\mathcal{C}_g^{r+u}$. This form is defined as follows. Suppose that the endpoints of e are $\bar{m}(i)$ and $\bar{m}(j)$. We define

$$h_e = p_{i,j}^* h \in \mathcal{R}^2(\mathcal{C}_g^{r+u}),$$

where $p_{i,j} : \mathcal{C}_g^{r+u} \to \mathcal{C}_g^2$ is the projection on the ith and jth coordinate. If e is a loop based at vertex $\bar{m}(i)$, then

$$h_e = p_{i,i}^* h = p_i^* \Delta^* h = p_i^* e^A,$$

where $p_i : \mathcal{C}_g^{r+u} \to \mathcal{C}_g$ is the projection on the ith coordinate, and $\Delta : \mathcal{C}_g \to \mathcal{C}_g^2$ is the diagonal morphism. Notice that h_e does not depend on the order of i and j as the form h is symmetric in the two coordinates of $\mathcal{C}_g^2$.

Now, we let μ_Γ denote the product of all these 2-forms:

$$\mu_\Gamma = \bigwedge_{e \in E} h_e \in \mathcal{R}^{2|E|}(\mathcal{C}_g^{r+u}).$$

This form depends on the choice of $\bar{m}$. However, the form obtained from a different choice of $\bar{m}$ only differs from μ_Γ by permutation of the last u coordinates of $\mathcal{C}_g^{r+u}$. Therefore, by Fubini's theorem, the fiber integral

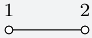
$$\alpha_\Gamma := \int_{p_{1,\ldots,r} : \mathcal{C}_g^{r+u} \to \mathcal{C}_g^r} \mu_\Gamma \in \mathcal{R}^{2(|E|-u)}(\mathcal{C}_g^r) \tag{4.4.1}$$

does not depend on the choice of $\bar{m}$.

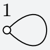
Definition 4.4.2. Let Γ be an r-marked graph. The form α_Γ on $\mathcal{C}_g^r$ defined in Equation 4.4.1 is *the (tautological) form associated to* Γ.

As the following examples show, many of the tautological differential forms we found before can be expressed as tautological forms associated to marked graphs.

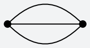
Example 4.4.3. Consider the unique 2-marked graph Γ with no unmarked vertices and a single edge between the two marked vertices. The associated 2-form α_Γ on $\mathcal{C}_g^2$ is h.

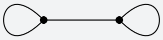
$\Gamma =$ 

Example 4.4.4. Consider the unique 1-marked graph Γ with no unmarked vertices and a single loop based at the unique vertex of Γ. The associated 2-form α_Γ on $\mathcal{C}_g$ is $\Delta^* h = e^A$.

$\Gamma =$ 

Example 4.4.5. Consider the two 0-marked graphs in the following picture.

$\Gamma_1 =$ 

$\Gamma_2 =$ 

The associated forms on $\mathcal{M}_g$ are

$$\alpha_{\Gamma_1} = \int_{\mathcal{C}_g^2/\mathcal{M}_g} h^3 =: \nu$$

and

$$\begin{aligned}
\alpha_{\Gamma_2} &= \int_{\mathcal{C}_g^2/\mathcal{M}_g} h \wedge p_1^* e^A \wedge p_2^* e^A \\
&= \int_{\mathcal{C}_g/\mathcal{M}_g} \int_{p_1 : \mathcal{C}_g^2 \to \mathcal{C}_g} h \wedge p_1^* e^A \wedge p_2^* e^A \\
&= \int_{\mathcal{C}_g/\mathcal{M}_g} \left(e^A \wedge \int_{p_1} h \wedge p_2^* e^A \right) \\
&= \int_{\mathcal{C}_g/\mathcal{M}_g} (e^A)^2 \\
&= e_1^A,
\end{aligned}$$

where we have used the projection formula and Lemma 4.3.4. We therefore see that the tautological form

$$\frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}} = \nu - e_1^A,$$

while not being associated to a graph itself, is in the linear span of forms on $\mathcal{M}_g$ associated to 0-marked graphs.

In the next section we will prove that, in fact, every tautological form on $\mathcal{C}_g^r$ can be written as the linear combination of forms associated to r-marked graphs.

4.5 Graph operations and tautological forms

In the previous section we introduced a combinatorial method of defining tautological forms on $\mathcal{C}_g^r$ for all $r \geq 0$ by associating them to r-marked graphs.

In this section we will study the various operations on marked graphs introduced in Chapter 3 and observe how the corresponding differential forms are affected. It turns out that these forms behave rather nicely with respect to pullbacks, pushforwards, and coproducts of marked graphs. By using this fact, we will be able to prove the following theorem.

Theorem 4.5.1. For every integer $r \geq 0$, the ring of tautological differential forms $\mathcal{R}^*(\mathcal{C}_g^r)$ is spanned as an $\mathbb{R}$-vector space by forms α_Γ associated to r-marked graphs Γ.

By Definition 4.3.1 it suffices to prove that the system of linear subspaces $S^*(\mathcal{C}_g^r) \subseteq \mathcal{R}^*(\mathcal{C}_g^r)$ generated by forms associated to r-marked graphs is a system of sub-$\mathbb{R}$-algebras (that is: closed under wedge products and containing 1), that the system is closed under pullbacks and fiber integrals, and that h is contained in $S^*(\mathcal{C}_g^2)$.

We start by proving that $S^*(\mathcal{C}_g^r) \subseteq \mathcal{R}^*(\mathcal{C}_g^r)$ is a subring for every $r \geq 0$. First of all, the form associated to the unique r-marked graph consisting of r vertices and no edges is 1. The following proposition implies that $S^*(\mathcal{C}_g^r)$ is closed under wedge products and therefore a subring of $\mathcal{R}^*(\mathcal{C}_g^r)$.

4

Proposition 4.5.2. Let $\Gamma = (V, E, m)$ and $\Gamma' = (V', E', m')$ be two r-marked graphs, and let α_Γ and $\alpha_{\Gamma'}$ be the associated tautological forms on $\mathcal{C}_g^r$. Then

$$\alpha_\Gamma \wedge \alpha_{\Gamma'} = \alpha_{\Gamma \sqcup_r \Gamma'}.$$

Proof. Assume that Γ and Γ' have respectively u and u' unmarked vertices. Choose bijective extensions

$$\bar{m} : \{1, \ldots, r+u\} \xrightarrow{\sim} V$$
$$\bar{m}' : \{1, \ldots, r+u'\} \xrightarrow{\sim} V'$$

of m and m'. Let $\phi : \{1, \ldots, r+u\} \to \{1, \ldots, r+u+u'\}$ be the inclusion, and define the map

$$\psi : \{1, \ldots, r+u'\} \to \{1, \ldots, r+u+u'\} : k \mapsto \begin{cases} k & \text{if } k \leq r \\ k+u & \text{if } k > r. \end{cases}$$

It follows that the diagram

$$\begin{tikzcd}
\{1, \ldots, r+u+u'\} & \{1, \ldots, r+u\} \arrow[l, "\phi"'] \\
\{1, \ldots, r+u'\} \arrow[u, "\psi"] & \{1, \ldots, r\} \arrow[l, hook'] \arrow[u, hook]
\end{tikzcd}$$

is a pushout diagram of sets, so we have the associated cartesian diagram of moduli stacks

$$\begin{tikzcd}
\mathcal{C}_g^{r+u+u'} \arrow[r, "p_{1,\dots,r+u}"] \arrow[d, "p_{1,\dots,r,r+u+1,\dots,r+u+u'}"'] & \mathcal{C}_g^{r+u} \arrow[d, "p_{1,\dots,r}"] \\
\mathcal{C}_g^{r+u'} \arrow[r, "p_{1,\dots,r}"] & \mathcal{C}_g^{r}.
\end{tikzcd}$$

Now let $\Gamma'' = (V'', E'', m'') = \Gamma \sqcup_r \Gamma'$. By the universal property of the pushout, we have an induced $r + u + u'$-marking

$$\bar{m}'' : \{1, \dots, r + u + u'\} \xrightarrow{\sim} V''$$

of the set of vertices V'' of Γ'' that extends m''. If $e \in E$ is an edge in Γ between vertices $\bar{m}(i)$ and $\bar{m}(j)$, then the corresponding edge in Γ'' has endpoints $\bar{m}''(\phi(i))$ and $\bar{m}''(\phi(j))$. Similarly, if $e \in E'$ is an edge in Γ' between vertices $\bar{m}'(i)$ and $\bar{m}'(j)$, then the corresponding edge in Γ'' has endpoints $\bar{m}''(\psi(i))$ and $\bar{m}''(\psi(j))$. It follows that

$$\begin{aligned}
\mu_{\Gamma''} &= \bigwedge_{e \in E''} h_e \\
&= \bigwedge_{e \in E} p^*_{1,\dots,r+u} h_e \wedge \bigwedge_{e \in E'} p^*_{1,\dots,r,r+u+1,\dots,r+u+u'} h_e \\
&= p^*_{1,\dots,r+u} \mu_\Gamma \wedge p^*_{1,\dots,r,r+u+1,\dots,r+u+u'} \mu_{\Gamma'}.
\end{aligned}$$

Using the base change formula 1.3.14 and the projection formula 1.3.1, we find that the fiber integral $\alpha_{\Gamma \sqcup_r \Gamma'}$ equals $\alpha_\Gamma \wedge \alpha_{\Gamma'}$. ☐

Next, we will show that the system of vector spaces $S^*(\mathcal{C}_g^r) \subseteq \mathcal{R}^*(\mathcal{C}_g^r)$ is closed under pullbacks along tautological morphisms. Let $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ be a tautological morphism, induced by a map $\phi : \{1, \dots, s\} \to \{1, \dots, r\}$. Recall from Chapter 3 that ϕ induces a pushforward operator $\phi_* : \mathcal{G}_r \to \mathcal{G}_s$ from r-marked graphs to s-marked graphs. The following proposition implies that the pullback map $f^{\phi,*}$ on differential forms is compatible with the pushforward map on graphs. From this one easily deduces that the system of forms $S^*(\mathcal{C}_g^r)$ is closed under pullbacks along tautological maps.

Proposition 4.5.3. Let $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ be the tautological morphism associated to a map $\phi : \{1, \dots, s\} \to \{1, \dots, r\}$. Suppose that $\alpha_\Gamma \in S^*(\mathcal{C}_g^s)$ is the form associated to an s-marked graph Γ. Then

$$f^{\phi,*} \alpha_\Gamma = \alpha_{\phi_* \Gamma}$$

with $\phi_* \Gamma$ the pushforward of Γ along ϕ.

Proof. The proof is similar to the proof of Proposition 4.5.2, so only a short sketch is given here. We extend the labeling on Γ to an $(s+u)$-labeling, with u the number of unmarked vertices of Γ. This induces an $(r + u)$-labeling of $\phi_* \Gamma$, and it follows that the pullback of μ_Γ along the induced map $\mathcal{C}_g^{r+u} \to \mathcal{C}_g^{s+u}$ equals $\mu_{\phi_*\Gamma}$. By the base change formula the desired result follows. ☐

Now, let $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ be a tautological submersion, associated to an injective map $\phi : \{1, \ldots, s\} \hookrightarrow \{1, \ldots, r\}$. In Chapter 3 we introduced a pullback map $\phi^* : \mathcal{G}_r \to \mathcal{G}_s$. The following proposition shows that, analogously to the pushforward map, the pullback map on graphs is compatible with the fiber integral map on differential forms. This implies that the system $S^*(\mathcal{C}_g^r) \subseteq \mathcal{R}^*(\mathcal{C}_g^r)$ is closed under fiber integrals.

Proposition 4.5.4. Let $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ be an injective map, and let $f^\phi : \mathcal{C}_g^r \to \mathcal{C}_g^s$ be the associated tautological submersion. Let $\Gamma \in \mathcal{G}_r$ be an r-marked graph, and let $\phi^*\Gamma$ be the s-marked graph induced by ϕ. Then

$$\int_{f^\phi} \alpha_\Gamma = \alpha_{\phi^*\Gamma}$$

Proof. Let u be the number of unmarked vertices in Γ. Extend the inclusion $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$ to a permutation $\{1, \ldots, r\} \to \{1, \ldots, r\}$, and then join this map with the identity on $\{r+1, \ldots, r+u\}$ to obtain a bijective map

$$\bar{\phi} : \{1, \ldots, r+u\} \xrightarrow{\sim} \{1, \ldots, r+u\}$$

that extends ϕ.

Moreover, choose a bijective extension $\bar{m} : \{1, \ldots, r+u\} \xrightarrow{\sim} V$ of the marking m of Γ. We immediately obtain an extension

$$\overline{m\phi} = \bar{m} \circ \bar{\phi} : \{1, \ldots, r+u\} \xrightarrow{\sim} V$$

of the marking $m\phi$ of the s-marked graph $\phi^*\Gamma = (V, E, m\phi)$. We have a commutative diagram of sets, inducing a commutative diagram of moduli stacks:

$$\begin{tikzcd}
\{1, \ldots, r+u\} & \{1, \ldots, r\} \arrow[l, hook', "\supseteq"'] \\
\{1, \ldots, r+u\} \arrow[u, "\bar{\phi}"'] & \{1, \ldots, s\} \arrow[l, hook', "\supseteq"'] \arrow[u, "\phi"']
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathcal{C}_g^{r+u} \arrow[r, "p_{1,\ldots,r}"] \arrow[d, "f^{\bar{\phi}}"] & \mathcal{C}_g^r \arrow[d, "f^\phi"] \\
\mathcal{C}_g^{r+u} \arrow[r, "p_{1,\ldots,s}"] & \mathcal{C}_g^s
\end{tikzcd}$$

If e is an edge in Γ with endpoints $\bar{m}(i), \bar{m}(j)$, then the corresponding edge ϕ^*e in $\phi^*\Gamma$ has endpoints $\overline{m\phi}(\bar{\phi}^{-1}(i))$ and $\overline{m\phi}(\bar{\phi}^{-1}(j))$. It follows that the corresponding 2-forms on $\mathcal{C}_g^{r+u}$ are related as follows:

$$h_e = f^{\bar{\phi},*} h_{\phi^*e}.$$

From this, we find that

$$\mu_\Gamma = f^{\bar{\phi},*} \mu_{\phi^*\Gamma},$$

so

$$\mu_{\phi^*\Gamma} = \int_{f^{\bar{\phi}}} \mu_\Gamma.$$

We therefore have:

$$\int_{f^\phi} \alpha_\Gamma = \int_{f^\phi} \int_{p_{1,\ldots,r}} \mu_\Gamma = \int_{p_{1,\ldots,s}} \int_{f^{\bar{\phi}}} \mu_\Gamma = \int_{p_{1,\ldots,s}} \mu_{\phi^*\Gamma} = \alpha_{\phi^*\Gamma},$$

proving the proposition. □

We have seen that differential forms associated to graphs are quite well-behaved with respect to the graph operations defined in Chapter 3. Using this, we can quite easily prove the main theorem of this section.

Proof of Theorem 4.5.1. By Proposition 4.5.2, and the fact that the form associated to the r-marked graph with no edges and no unmarked vertices equals 1, we find that the subspaces $S^*(\mathcal{C}_g^r) \subseteq \mathcal{R}^*(\mathcal{C}_g^r)$ are, in fact, sub-$\mathbb{R}$-algebras. Propositions 4.5.3 and 4.5.4 show that the system of subspaces $S^*(\mathcal{C}_g^r) \subseteq \mathcal{R}^*(\mathcal{C}_g^r)$ is closed under taking pullbacks along tautological morphisms and fiber integrals along tautological submersions. Example 4.4.3 shows that h is an element of $S^2(\mathcal{C}_g^2)$.

But as the system of rings $\mathcal{R}^*(\mathcal{C}_g^r)$ is defined in Definition 4.3.1 to be the smallest system that satisfies these properties, we find that the two systems must be equal. □

4.6 Graph contractions and tautological forms

In the last section, we proved that every tautological form is a linear combination of tautological forms associated to graphs. We did so by observing the behavior of the resulting tautological forms when manipulating the marked graphs using the pushforward, pullback, and gluing operations in Chapter 3. In Section 3.6 we defined contraction operations on r-marked graphs. In this section we will show that these contractions are well-behaved with respect to taking associated tautological forms. This will allow us to prove the following theorem.

Theorem 4.6.1. For all $r \geq 0$ and $g \geq 2$, the ring of tautological forms $\mathcal{R}^*(\mathcal{C}_g^r)$ is finite-dimensional.

In the following proposition, we consider the various graph contraction operations defined in Chapter 3, and see how contracting vertices on an r-marked graph Γ influences the associated tautological form α_Γ. The proposition will be proved at the end of this section.

Proposition 4.6.2. Let $\Gamma = (V, E, m)$ be an r-marked graph, and suppose that Γ has an unmarked vertex v, such that either $\deg(v) \leq 2$, or $\deg(v) = 3$ and v is incident to precisely two edges. Let Γ' be the graph obtained from Γ by contracting v.

0. If $\deg v = 0$, then $\alpha_\Gamma = 0$.
1. If $\deg v = 1$, then $\alpha_\Gamma = \alpha_{\Gamma'}$.

2a. Suppose that $\deg v = 2$ and that v has two distinct neighbors $w \neq w'$. Then

$\alpha_\Gamma = \alpha_{\Gamma'}$.

2b. Suppose that $\deg v = 2$ and that v has a single neighbor $w \neq v$. Then $\alpha_\Gamma = \alpha_{\Gamma'}$.

2c. Suppose that $\deg v = 2$ and that v is its own neighbor; that is: there is a loop based at v. Then $\alpha_\Gamma = (2-2g)\alpha_{\Gamma'}$.

3. Suppose that $\deg v = 3$ and that v is incident to precisely two edges. Then $\alpha_\Gamma = \alpha_{\Gamma'}$.

Proposition 4.6.2 shows that we can contract every r-marked graph to a contracted r-marked graph while leaving the resulting tautological form the same up to multiplication by zero or a power of $(2-2g)$. Therefore, we find that the ring of tautological forms $\mathcal{R}^*(\mathcal{C}_g^r)$ is the linear span of the tautological forms associated to contracted r-marked graphs.

Suppose that Γ is an r-marked graph with u unmarked vertices and e edges. The Euler characteristic of Γ is $\chi(\Gamma) = r+u-e$. After extending the marking of Γ to an $(r+u)$-marking, we obtain the form μ_Γ that lives on $\mathcal{C}_g^{r+u}$ and has degree $2e$. Now α_Γ is the fiber integral of μ_Γ along the projection $\mathcal{C}_g^{r+u} \to \mathcal{C}_g^r$, whose fibers are of real dimension $2u$, and hence the degree of α_Γ is $2e-2u = 2r-2\chi(\Gamma)$. We obtain the following.

Lemma 4.6.3. Let $d \geq 0$ and $r \geq 0$ be integers. The space $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ of tautological forms of degree $2d$ on $\mathcal{C}_g^r$ is the linear span of the forms α_Γ associated to contracted r-marked graphs Γ with Euler characteristic $\chi(\Gamma) = r-d$. $\square$

In Theorem 3.7.1, we proved that there are (up to isomorphism) only finitely many contracted r-marked graphs of any given characteristic $\chi \in \mathbb{Z}$. By combining Lemma 4.6.3 with Theorem 3.7.1, we obtain the following.

Theorem 4.6.4. Let $g \geq 2$. For all integers $r \geq 0$ and $d \geq 0$, the space $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ of tautological forms of degree $2d$ on $\mathcal{C}_g^r$ is finite-dimensional. More precisely: the space $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ is spanned by forms α_Γ, where Γ ranges over all contracted r-marked graphs of characteristic $r-d$. These graphs have at most $2d$ unmarked vertices, and there are only finitely many such graphs up to isomorphism. $\square$

The main theorem of this section is now a simple consequence of the previous theorem.

Proof of Theorem 4.6.1. Recall from Section 2.5 that there exists an inclusion $A^*(\mathcal{C}_g^r) \to A^*(\mathcal{X}_g^r)$ where $\mathcal{X}_g \to \mathcal{T}_g$ is the universal family of genus g curves with Teichmüller structure and $\mathcal{X}_g^r$ denotes the r-fold fiber product of $\mathcal{X}_g$ over $\mathcal{T}_g$. As $\mathcal{X}_g^r$ is a manifold of (real) dimension $6g-6+2r$, it follows that $A^d(\mathcal{X}_g^r)$ is zero for all $d > 2r+6g-6$, and the same is true for $A^d(\mathcal{C}_g^r)$ and hence for $\mathcal{R}^d(\mathcal{C}_g^r)$. Moreover, the odd-degree subspaces $\mathcal{R}^{2d+1}(\mathcal{C}_g^r)$ are zero. Therefore

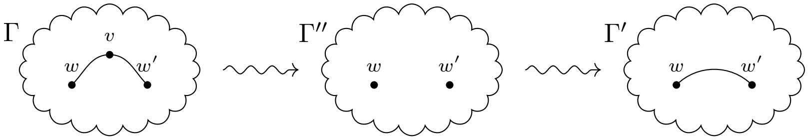
$$\mathcal{R}^*(\mathcal{C}_g^r) = \bigoplus_{d\geq 0} \mathcal{R}^*(\mathcal{C}_g^r) = \bigoplus_{d=0}^{3g-3+r} \mathcal{R}^{2d}(\mathcal{C}_g^r)$$

is a direct sum of finitely many finite-dimensional subspaces, and therefore it is itself finite-dimensional. ☐

We devote the remainder of this section to proving Proposition 4.6.2. The proof is merely technical, and does not introduce any new concepts.

Proof of Proposition 4.6.2. Let $\Gamma = (V, E, m)$ be an r-marked graph, and let v be an unmarked vertex of degree ≤ 2, or an unmarked vertex of degree ≤ 3 with a loop. Define a graph Γ'' by removing v, and all edges emanating from v, from Γ. Moreover, we have the graph Γ' that is obtained from Γ by contracting v.

The graph Γ'' represents an 'intermediate step' in obtaining Γ' from Γ. The following picture describes the situation in the case where v has two distinct neighbors.



Let $u \geq 0$ be such that Γ has $u+1$ unmarked points. Fix an extension of m to an $(r+u+1)$-marking

$$\bar{m} : \{1, \ldots, r+u+1\} \xrightarrow{\sim} V,$$

such that $\bar{m}(r+u+1) = v$.

Restricting $\bar{m}$ to $\{1, \ldots, r+u\}$ induces an $(r+u)$-marking on Γ' and Γ'' that extends the r-marking on these graphs. We obtain differential forms μ_Γ, $\mu_{\Gamma'}$, and $\mu_{\Gamma''}$ that live on $\mathcal{C}_g^{r+u+1}$, $\mathcal{C}_g^{r+u}$, and $\mathcal{C}_g^{r+u}$, respectively.

The inclusions $\{1, \ldots, r\} \subseteq \{1, \ldots, r+u\} \subseteq \{1, \ldots, r+u+1\}$ induce tautological submersions

$$\begin{tikzcd}
\mathcal{C}_g^{r+u+1} \arrow[r, "q"] \arrow[dr, "pq"'] & \mathcal{C}_g^{r+u} \arrow[d, "p"] \\
& \mathcal{C}_g^{r}
\end{tikzcd}$$

We have

$$\alpha_\Gamma = \int_{pq} \mu_\Gamma \quad \text{and} \quad \alpha_{\Gamma'} = \int_p \mu_{\Gamma'}$$

If we can prove that $\int_q \mu_\Gamma = 0$ in case 0, $\int_q \mu_\Gamma = \mu_{\Gamma'}$ in cases 1, 2a, 2b, and 3, and $\int_q \mu_\Gamma = (2-2g)\mu_{\Gamma'}$ in case 2c, we are done.

0. Suppose v has degree 0. The set of edges of Γ is equal to the set of edges of Γ', so we obtain

   $$\mu_\Gamma = q^* \mu_{\Gamma'}.$$

   Taking fiber integrals and applying the projection formula yields:

   $$\int_q \mu_\Gamma = \mu_{\Gamma'} \int_q 1 = 0,$$

   and we find that $\alpha_\Gamma = 0$.

1. Suppose v has degree 1; let $i \in \{1, \ldots, r+u\}$ be such that $\bar{m}(i)$ is the neighbor of v. The graph Γ is obtained from Γ' by adding the vertex v and the edge between v and $\bar{m}(i)$. We therefore have:

$$\mu_\Gamma = q^* \mu_{\Gamma'} \wedge p_{i,r+u+1}^* h,$$

so

$$\int_q \mu_\Gamma = \mu_{\Gamma'} \wedge \int_q p_{i,r+u+1}^* h.$$

By using the base change formula with the cartesian diagram

$$\begin{tikzcd}
\mathcal{C}_g^{r+u+1} \arrow[r, "p_{i,r+u+1}"] \arrow[d, "q"] \arrow[dr, phantom, "\square"] & \mathcal{C}_g^2 \arrow[d, "p_1"] \\
\mathcal{C}_g^{r+u} \arrow[r, "p_i"'] & \mathcal{C}_g,
\end{tikzcd}$$

we find:

$$\int_q p_{i,r+u+1}^* h = p_i^* \int_{p_1} h = 1,$$

where the latter equality follows from Lemma 4.3.3. This shows that $\int_q \mu_\Gamma = \mu_{\Gamma'}$, so $\alpha_\Gamma = \alpha_{\Gamma'}$.

2a. Suppose v has degree 2, and that v has two distinct neighbors w and w'. Let $i, j \in \{1, \ldots, r+u\}$ be such that $\bar{m}(i) = w$ and $\bar{m}(j) = w'$. In this case, we find

$$\mu_\Gamma = \mu_{\Gamma''} \wedge p_{i,r+u+1}^* h \wedge p_{j,r+u+1}^* h,$$

and

$$\mu_{\Gamma'} = \mu_{\Gamma''} \wedge p_{i,j}^* h.$$

In this case, another application of the base change formula, together with the identity of forms

$$\int_{p_{12}} p_{13}^* h \wedge p_{23}^* h = h$$

from Lemma 4.3.5 shows that $\int_q \mu_\Gamma = \mu_{\Gamma'}$, and hence $\alpha_\Gamma = \alpha_{\Gamma'}$.

2b. The proof in this case is very similar to the proofs for cases 1 and 2. In this case, we use the identity

$$\int_{p_1} h^2 = e^A$$

from Lemma 4.3.4.

2c. Again, the proof of this case is similar to that of the previous cases. The identity used here is

$$\int_{\mathcal{C}_g/\mathcal{M}_g} e^A = (2-2g),$$

see Lemma 4.3.2.

3. Finally, the proof in case 3 is analogous to that of earlier cases, where we use the identity
$$\int_{p_1} h \wedge p_2^* e^A = e^A$$
from Lemma 4.3.4.

□

4.7 The ring of tautological forms as a quotient algebra

In the previous sections we proved that the ring of tautological forms $\mathcal{R}^*(\mathcal{C}_g^r)$ is the linear span of forms associated to r-marked graphs. In this section we will exploit this and show that we can write the ring of tautological forms $\mathcal{R}^*(\mathcal{C}_g^r)$ as a quotient algebra of a graded $\mathbb{R}$-algebra whose summands are effectively computable.

Recall from Chapter 3 that $\mathrm{G}(r)$ denotes the set of isomorphism classes of r-marked graphs. The coproduct $\sqcup_r$ on the category $\mathcal{G}_r$ of r-marked graphs induces a binary operator on $\mathrm{G}(r)$, which gives $\mathrm{G}(r)$ the structure of a commutative monoid. There is a homomorphism of monoids

$$\bar{\chi}_r : \mathrm{G}(r) \to \mathbb{Z} : \Gamma \mapsto r - \chi(\Gamma).$$

If $d \in \mathbb{Z}$ is an integer, the inverse image $\bar{\chi}_r^{-1}(d)$ is the set

$$\mathrm{G}(r, r-d) = \{r\text{-marked graphs of characteristic } r-d\}/\cong.$$

The *monoid ring* of $\mathrm{G}(r)$ over $\mathbb{R}$ is the $\mathbb{R}$-algebra

$$\mathbb{R}[\mathrm{G}(r)]$$

that has as the underlying $\mathbb{R}$-module the vector space with the elements of $\mathrm{G}(r)$ as its basis, and whose multiplication is defined uniquely by demanding it extends the binary operator on $\mathrm{G}(r)$, where we view $\mathrm{G}(r)$ as a subset of $\mathbb{R}[\mathrm{G}(r)]$ via the map $\Gamma \mapsto 1 \cdot \Gamma$. The homomorphism $\bar{\chi}_r : \mathrm{G}(r) \to \mathbb{Z}$ and the corresponding partition of $\mathrm{G}(r)$ induce a grading on $\mathbb{R}[\mathrm{G}(r)]$ whose degree d summand is spanned by the graphs of characteristic $r-d$:

$$\mathbb{R}[\mathrm{G}(r)] = \bigoplus_{d \in \mathbb{Z}} \mathbb{R}[\mathrm{G}(r, r-d)].$$

The method described in Section 4.4 of taking a graph $\Gamma \in \mathrm{G}(r)$ and assigning to it a tautological form $\alpha_\Gamma \in \mathcal{R}^*(\mathcal{C}_g^r)$ induces a map of sets $\mathrm{G}(r) \to \mathcal{R}^*(\mathcal{C}_g^r)$. This map is in fact a homomorphism of monoids by Proposition 4.5.2, where the monoid structure on $\mathcal{R}^*(\mathcal{C}_g^r)$ is given by the wedge product. This monoid homomorphism induces a homomorphism of $\mathbb{R}$-algebras

$$\alpha : \mathbb{R}[G(r)] \to \mathcal{R}^*(\mathcal{C}_g^r).$$

By Theorem 4.5.1 it holds that this homomorphism is surjective.

If Γ is an r-marked graph of characteristic $r-d$, the corresponding form α_Γ is of degree $2d$, and hence the above $\mathbb{R}$-algebra homomorphism is in fact a homomorphism of graded $\mathbb{R}$-algebras

$$\bigoplus_{d\in\mathbb{Z}} \mathbb{R}[\mathrm{G}(r,r-d)] \to \bigoplus_{d\in\mathbb{Z}} \mathcal{R}^{2d}(\mathcal{C}_g^r).$$

Next, consider the submonoid $\mathrm{CG}(r) \subseteq \mathrm{G}(r)$ and the subsets $\mathrm{CG}(r,r-d) \subseteq \mathrm{G}(r,r-d)$ consisting of graphs that are contracted. The inclusion map $\mathrm{CG}(r) \to \mathrm{G}(r)$ induces a homomorphism of (graded) $\mathbb{R}$-algebras

$$\mathbb{R}[\mathrm{CG}(r)] \to \mathbb{R}[\mathrm{G}(r)],$$

and the composition of this map with α yields another graded homomorphism

$$\alpha' : \mathbb{R}[\mathrm{CG}(r)] \to \mathcal{R}^{2*}(\mathcal{C}_g^r).$$

Theorem 4.6.4 says that this homomorphism is surjective.

By Theorem 3.8.1 the set $\mathrm{CG}(r,r-d)$ is empty for all $d<0$. For each $d\geq 0$ the set $\mathrm{CG}(r,r-d)$ is effectively computable by using the algorithm found in Section 3.7. Moreover, for $d>r+3g-3$ the space $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ is trivial, so for these d the degree d summand of $\mathbb{R}[\mathrm{CG}(r,r-d)]$ is contained in the kernel of α'. It follows that, in order to compute $\mathcal{R}^*(\mathcal{C}_g^r)$, we need to compute the kernel of the linear map

$$\mathbb{R}[\mathrm{CG}(r,r-d)] \to \mathcal{R}^{2d}(\mathcal{C}_g^r)$$

for all $0\leq d\leq r+3g-3$. In any case, we obtain the following.

Theorem 4.7.1. The graded $\mathbb{R}$-algebra $\mathcal{R}^{2*}(\mathcal{C}_g^r)$ is a quotient of the monoid ring $\mathbb{R}[\mathrm{CG}(r)]$. More precisely, it is a quotient of the quotient ring

$$\frac{\mathbb{R}[\mathrm{CG}(r)]}{\mathbb{R}[\mathrm{CG}(r)]_{>(r+3g-3)}} = \frac{\mathbb{R}[\mathrm{CG}(r)]}{\bigoplus_{d>r+3g-3}\mathbb{R}[\mathrm{CG}(r,r-d)]}.$$

and that quotient ring is effectively computable; it is isomorphic as a vector space to

$$\bigoplus_{d=0}^{r+3g-3} \mathbb{R}[\mathrm{CG}(r,r-d)] \qquad \square$$

The inclusion $\mathrm{CG}(r) \to \mathrm{G}(r)$ has a retraction $\varrho : \mathrm{G}(r) \to \mathrm{CG}(r)$ which is the contraction map. Therefore, the induced homomorphism $\mathbb{R}[\mathrm{G}(r)] \to \mathbb{R}[\mathrm{CG}(r)]$ is a retraction of the inclusion $\mathbb{R}[\mathrm{CG}(r)] \to \mathbb{R}[\mathrm{G}(r)]$. However, it turns out that this retraction is not the right retraction for our purposes: this retraction is incompatible with the homomorphisms from these monoid rings to the ring of tautological forms. For instance, if Γ denotes the 0-marked graph with one vertex and one loop, then the associated form α_Γ is the constant function $(2-2g)$. The contracted graph $\varrho(\Gamma)$ is the empty graph, and the associated form is the constant function 1; we therefore see that $\alpha_\Gamma \neq \alpha_{\varrho(\Gamma)}$. It is more natural to define an $\mathbb{R}$-algebra homomorphism

$$\tilde{\varrho}_g : \mathbb{R}[\mathrm{G}(r)] \to \mathbb{R}[\mathrm{CG}(r)],$$

that depends on g, as follows.

Let Γ be an r-marked graph, and let $\varrho(\Gamma)$ be the corresponding contracted r-marked graph. Define the integer $\lambda_{\Gamma,g}$ as:

$$\lambda_{\Gamma,g} = 0^a \cdot (2-2g)^b,$$

where a and b denote the number of contractions of type 0 and 2c, respectively, in the contraction procedure. Equivalently, a and b equal the number of connected components of Γ, without marked vertices, of characteristic 1 and 0, respectively. It follows from Proposition 4.6.2 that

$$\alpha_\Gamma = \lambda_{\Gamma,g} \cdot \alpha_{\varrho(\Gamma)}.$$

We define the $\mathbb{R}$-algebra homomorphism $\tilde{\varrho}_g : \mathbb{R}[\mathrm{G}(r)] \to \mathbb{R}[\mathrm{CG}(r)]$ by setting

$$\tilde{\varrho}_g(\Gamma) = \lambda_{\Gamma,g} \cdot \varrho(\Gamma) \quad \text{for all } \Gamma \in \mathrm{G}(r).$$

As $\lambda_{\Gamma,g} = 1$ for all contracted graphs, it follows that $\tilde{\varrho}_g$ is a retraction of the inclusion map $\mathbb{R}[\mathrm{CG}(r)] \to \mathbb{R}[\mathrm{G}(r)]$. Moreover, the following diagram commutes.

$$\begin{tikzcd}
\mathbb{R}[\mathrm{G}(r)] \arrow[dr, two heads] \arrow[dd, two heads, "\tilde{\varrho}_g"'] & \\
& \mathcal{R}^{2*}(\mathcal{C}_g^r) \\
\mathbb{R}[\mathrm{CG}(r)] \arrow[ur, two heads] &
\end{tikzcd} \tag{4.7.2}$$

4.8 Tautological 2-forms

In this section we give a description of the vector spaces $\mathcal{R}^2(\mathcal{C}_g^r)$ of tautological two-forms on the spaces $\mathcal{C}_g^r$ for all $r \geq 0$. Recall that we have seen some examples of these 2-forms already: on $\mathcal{C}_g^2$ we have the 2-form h associated to the diagonal, on $\mathcal{C}_g$ we have $e^A = \Delta^* h$ associated to the tangent bundle, and on $\mathcal{M}_g$ we found two more 2-forms

$$e_1^A := \int_{\mathcal{C}_g/\mathcal{M}_g} (e^A)^2 \quad \text{and} \quad \nu := \int_{\mathcal{C}_g^2/\mathcal{M}_g} h^3.$$

We will prove that these 2-forms are 'all there is': the tautological ring $\mathcal{R}^2(\mathcal{C}_g^r)$ is spanned by pullbacks of h, e^A, e_1^A, and ν along tautological submersions.

Let $r \geq 0$ be an integer. We wish to compute generators for $\mathcal{R}^2(\mathcal{C}_g^r)$. By Theorem 4.6.4, we find that this space is spanned by forms α_Γ, where Γ ranges over all contracted r-marked graphs of characteristic $r-1$. In Example 3.7.2 we have computed the set $\mathrm{CG}(r, r-1)$. We found the following graphs:

- Graphs Γ with r marked vertices, no unmarked vertices, and a single edge. If this edge is a loop based at vertex i then the associated form is

  $$\alpha_\Gamma = p_i^* e^A.$$

  If the edge is not a loop, and its endpoints are vertices i and j, then the associated form is

  $$\alpha_\Gamma = p_{ij}^* h.$$

- The graph Γ with r marked vertices, one unmarked vertex, and two loops based at the unmarked vertex. The associated form is

  $$\alpha_\Gamma = \int_{p_{1,\dots,r}:\mathcal{C}_g^{r+1}\to\mathcal{C}_g^r} p_{r+1}^*(e^A)^2 = e_1^A$$

  by the base change formula. Note the slight abuse of notation here: we write e_1^A for the pullback of e_1^A along the tautological morphism $\mathcal{C}_g^r \to \mathcal{M}_g$.
- The graph Γ with r marked vertices, two unmarked vertices, and three edges between the unmarked vertices. By using the base change formula we obtain

  $$\alpha_\Gamma = \int_{p_{1,\dots,r}:\mathcal{C}_g^{r+2}\to\mathcal{C}_g^r} p_{r+1,r+2}^* h^3 = \nu$$

  where we again abuse the notation by writing ν for the pullback of ν along $\mathcal{C}_g^r \to \mathcal{M}_g$.

We find that $\mathcal{R}^2(\mathcal{C}_g^r)$ is spanned by the following collection of 2-forms:

$$\{p_{ij}^* h : 1 \le i < j \le r\} \cup \{p_i^* e^A : 1 \le i \le r\} \cup \{e_1^A, \nu\}. \tag{4.8.1}$$

In the remainder of this section, we prove that, in fact, these 2-forms form a basis of $\mathcal{R}^2(\mathcal{C}_g^r)$ if $g > 2$, and there is only one relation among these forms if $g = 2$.

Theorem 4.8.2. For all $g \ge 2$ and $r \ge 0$, we have

$$\dim \mathcal{R}^2(\mathcal{C}_g^r) = \tfrac{1}{2} r(r+1) + 2 - \varepsilon(g),$$

where

$$\varepsilon(g) = \begin{cases} 1 & \text{if } g = 2 \\ 0 & \text{if } g \ge 3. \end{cases}$$

If $g \ge 3$ a basis is given by the 2-forms

$$\{p_{ij}^* h : 1 \le i < j \le r\} \cup \{p_i^* e^A : 1 \le i \le r\} \cup \{e_1^A, \nu\}.$$

If $g = 2$ a basis is given by the 2-forms

$$\{p_{ij}^* h : 1 \le i < j \le r\} \cup \{p_i^* e^A : 1 \le i \le r\} \cup \{e_1^A\},$$

and e_1^A and ν are linearly dependent: we have

$$-8\nu - 12 e_1^A = 0.$$

We will prove this theorem by induction on r. We start with the following proposition.

Proposition 4.8.3. If $g = 2$, then $\mathcal{R}^2(\mathcal{M}_g)$ is one-dimensional, and spanned by e_1^A. If $g \geq 3$, then $\mathcal{R}^2(\mathcal{M}_g)$ is two-dimensional, and spanned by e_1^A and ν.

Proof. Recall that the following identity holds:

$$\nu - e_1^A = \frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}}.$$

Before stating Theorem 4.8.2, we showed for all $r \geq 0$ that $\mathcal{R}^2(\mathcal{C}_g^r)$ is spanned by the forms listed in Equation 4.8.1. In particular, $\mathcal{R}^2(\mathcal{M}_g)$ is spanned by ν and e_1^A. Therefore, the dimension of $\mathcal{R}^2(\mathcal{M}_g)$ is at most two.

Suppose, first, that $g = 2$. In Example 4.10.5, which does not depend on any of the material treated in this section, we obtain the relation

$$-8\nu - 12e_1^A = 0.$$

Moreover, the real 2-form

$$\frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}}$$

is nonzero; see [DG14]. We conclude that $\mathcal{R}^2(\mathcal{M}_2)$ is one-dimensional.

Now, suppose that $g \geq 3$. By observing the asymptotic behavior of φ around the boundary of $\mathcal{M}_g$ studied in [dJon14], we find in particular that φ is not constant. Using [Kaw09, Lemma 8.1] we deduce:

$$\frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}} \neq 0$$

Moreover, the cohomology class κ_1 associated to e_1^A does not vanish. Indeed, one can show (see, for instance, [Mum83]) that $\kappa_1 = 12\lambda_1$ with λ_1 the first Chern class of the Hodge bundle on $\mathcal{M}_g$, and in [AC87] it is proved that λ_1 freely generates the Picard group of $\mathcal{M}_g$, and is in particular not torsion. Consequently, e_1^A is not an exact form; we find therefore that e_1^A and $\frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}}$ are linearly independent. ☐

Proof of Theorem 4.8.2. The case $r = 0$ is proved in Proposition 4.8.3. For the case $r = 1$: by Lemma 2.5.4 the morphism $p : \mathcal{C}_g \to \mathcal{M}_g$ induces an inclusion $p^* : \mathcal{R}^2(\mathcal{M}_g) \to \mathcal{R}^2(\mathcal{C}_g)$. Moreover, forms pulled back from $\mathcal{M}_g$ are in the kernel of the fiber integral along p: for each $\alpha \in A^*(\mathcal{M}_g)$ we have by the projection formula:

$$\int_p p^*\alpha = \alpha \cdot \int_p 1 = 0$$

As

$$\int_p e^A = (2 - 2g) \neq 0,$$

we find that e^A is not an element of $p^*\mathcal{R}^2(\mathcal{M}_g)$. As $\mathcal{R}^2(\mathcal{C}_g)$ is spanned by the forms e^A, e_1^A, and ν, we obtain:

$$\dim \mathcal{R}^2(\mathcal{C}_g) = \dim \mathcal{R}^2(\mathcal{M}_g) + 1 = 3 - \varepsilon(g).$$

Now let $r \geq 2$, and assume that

$$\dim \mathcal{R}^2(\mathcal{C}_g^s) = \tfrac{1}{2}s(s+1) + 2 - \varepsilon(g)$$

for all $0 \leq s < r$. Consider the following three tautological morphisms:

$$\begin{aligned} p_{(r)} &: \mathcal{C}_g^r \to \mathcal{C}_g^{r-1} : (x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_{r-1}); \\ p_{(r-1)} &: \mathcal{C}_g^r \to \mathcal{C}_g^{r-1} : (x_1, \ldots, x_r) \mapsto (x_1, \ldots, x_{r-2}, x_r); \\ q_{(r-1)} &: \mathcal{C}_g^{r-1} \to \mathcal{C}_g^{r-2} : (x_1, \ldots, x_{r-1}) \mapsto (x_1, \ldots, x_{r-2}). \end{aligned}$$

We have a cartesian square

$$\begin{tikzcd} \mathcal{C}_g^r \arrow[r, "p_{(r)}"] \arrow[d, "p_{(r-1)}"'] \arrow[dr, phantom, "\square"] & \mathcal{C}_g^{r-1} \arrow[d, "q_{(r-1)}"] \\ \mathcal{C}_g^{r-1} \arrow[r, "q_{(r-1)}"] & \mathcal{C}_g^{r-2}. \end{tikzcd}$$

These maps induce linear subspaces $W_1 := \operatorname{Im} p_{(r)}^*$, $W_2 := \operatorname{Im} p_{(r-1)}^*$, and $W_{12} := W_1 \cap W_2$ of $\mathcal{R}^2(\mathcal{C}_g^r)$. The forms e_1^A, ν, $p_i^* e^A$, and $p_{ij}^* h$, (possibly) except for the form $p_{r-1,r}^* h$, all lie in W_1 or W_2. It follows that

$$\mathcal{R}^2(\mathcal{C}_g^r) = (W_1 + W_2) + \mathbb{R} \cdot p_{r-1,r}^* h.$$

Obviously the pullback of each form on $\mathcal{C}_g^{r-2}$ along the composition $q_{(r-1)} \circ p_{(r-1)} = q_{(r-1)} \circ p_{(r)}$ is an element of W_{12}. Conversely, we claim that each form in W_{12} is the pullback along this composition of some form on $\mathcal{C}_g^{r-2}$. Indeed, let $\alpha \in W_{12}$ be any form; we may write $\alpha = p_{(r)}^* \beta = p_{(r-1)}^* \gamma$ for forms $\beta, \gamma \in \mathcal{R}^2(\mathcal{C}_g^{r-1})$. Let $\mu \in \mathcal{R}^2(\mathcal{C}_g)$ be the 2-form given by $\mu = e^A/(2-2g)$; it follows that $\int_{\mathcal{C}_g/\mathcal{M}_g} \mu = 1$, and by the base change formula we obtain

$$\int_{p_{(r)}} p_r^* \mu = 1.$$

We then find by repeatedly using the base change formula and the projection formula:

$$\begin{aligned} \beta &= \beta \wedge \int_{p_{(r)}} p_r^* \mu \\ &= \int_{p_{(r)}} p_{(r)}^* \beta \wedge p_r^* \mu \\ &= \int_{p_{(r)}} p_{(r-1)}^* \gamma \wedge p_r^* \mu \\ &= \int_{p_{(r)}} p_{(r-1)}^* (\gamma \wedge p_{r-1}^* \mu) \\ &= q_{(r-1)}^* \int_{q_{(r-1)}} \gamma \wedge p_{r-1}^* \mu, \end{aligned}$$

and therefore

$$\alpha = p_{(r)}^* \beta = p_{(r)}^* q_{(r-1)}^* \int_{q_{(r-1)}} \gamma \wedge p_{r-1}^* \mu,$$

which proves our claim.

As pullbacks along tautological submersions are injective, we obtain the following equalities from the induction hypothesis:

$$\begin{aligned} \dim W_1 = \dim W_2 = \dim \mathcal{R}^2(\mathcal{C}_g^{r-1}) &= \tfrac{1}{2}r^2 - \tfrac{1}{2}r + 2 - \varepsilon(g) \\ \dim W_{12} = \dim \operatorname{Im}(p_{(r)}^* \circ q_{(r-1)}^*) = \dim \mathcal{R}^2(\mathcal{C}_g^{r-2}) &= \tfrac{1}{2}r^2 - \tfrac{3}{2}r + 3 - \varepsilon(g) \\ \dim(W_1 + W_2) = \dim W_1 + \dim W_2 - \dim W_{12} &= \tfrac{1}{2}r^2 + \tfrac{1}{2}r + 1 - \varepsilon(g). \end{aligned}$$

If we can prove that $p_{r-1,r}^* h \notin W_1 + W_2$ then $\dim \mathcal{R}^2(\mathcal{C}_g^r) = \frac{1}{2}r^2 + \frac{1}{2}r + 2 - \varepsilon(g)$ and we are done. Suppose, therefore, that $p_{r-1,r}^* h \in W_1 + W_2$; we can write $p_{r-1,r}^* h = p_{(r)}^* \alpha + p_{(r-1)}^* \beta$ for some 2-forms α, β on $\mathcal{C}_g^{r-1}$. As h is symmetric, we may even assume with no loss of generality that $\alpha = \beta$:

$$p_{r-1,r}^* h = p_{(r)}^* \alpha + p_{(r-1)}^* \alpha.$$

Consider the map

$$f : \mathcal{C}_g^{r-1} \to \mathcal{C}_g^r : (x_1, \ldots, x_{r-1}) \mapsto (x_1, \ldots, x_{r-1}, x_{r-1});$$

this map is a section of both $p_{(r)}$ and $p_{(r-1)}$ and fits in a cartesian diagram

$$\begin{tikzcd} \mathcal{C}_g^{r-1} \arrow[r, "f"] \arrow[d, "p_{r-1}"] & \mathcal{C}_g^r \arrow[d, "p_{r-1,r}"] \\ \mathcal{C}_g \arrow[r, "\Delta"] & \mathcal{C}_g^2. \end{tikzcd}$$

We then find:

$$p_{r-1}^* e^A = p_{r-1}^* \Delta^* h = f^* p_{r-1,r}^* h = 2\alpha;$$

so $\alpha = \frac{1}{2} p_{r-1}^* e^A$, and

$$p_{r-1,r}^* h = \tfrac{1}{2} p_{r-1}^* e^A + \tfrac{1}{2} p_r^* e^A \in \mathcal{R}^2(\mathcal{C}_g^r).$$

Integration along the fibers of the morphism $p_{(r)} : \mathcal{C}_g^r \to \mathcal{C}_g^{r-1}$ then yields:

$$1 = \int_{p_{(r)}} p_{r-1,r}^* h = \int_{p_{(r)}} \tfrac{1}{2}(p_{r-1}^* e^A + p_r^* e^A) = 0 + \tfrac{1}{2}(2 - 2g),$$

which contradicts with our assumption that $g \geq 2$. We conclude that $p_{r-1,r}^* h$ is not in the span of the subspaces W_1 and W_2, so we find:

$$\dim \mathcal{R}^2(\mathcal{C}_g^r) = \dim(W_1 + W_2) + 1 = \tfrac{1}{2}r^2 + \tfrac{1}{2}r + 2 - \varepsilon(g).$$

The theorem follows by induction. □

Let us return to the discussion we started in Section 4.3.

Corollary 4.8.4. The subspace of exact 2-forms $I^2(\mathcal{M}_g) \subseteq \mathcal{R}^2(\mathcal{M}_g)$ is one-dimensional. It is spanned by the form

$$\frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}} = \nu - e_1^A$$

where $\varphi : \mathcal{M}_g \to \mathbb{R}$ denotes the Kawazumi–Zhang invariant.

Proof. The cohomology classes of ν and e_1^A are equal: if $p : \mathcal{C}_g \to \mathcal{M}_g$, $q : \mathcal{C}_g^2 \to \mathcal{M}_g$, and $p_1, p_2 : \mathcal{C}_g^2 \to \mathcal{C}_g$ denote the tautological submersions, then

$$[\nu] = q_*(\Delta^3) = q_*(p_1^* K^2 \cdot \Delta) = p_*(K^2 \cdot p_{1,*}\Delta) = p_* K^2 = \kappa_1 = [e_1^A] \in RH^2(\mathcal{M}_g).$$

It follows that $\nu - e_1^A$ is exact. It is moreover a nonzero form, as we saw in the proof of Proposition 4.8.3, so the dimension of $I^2(\mathcal{M}_g)$ is positive. If $g = 2$ this concludes our proof, since $\dim \mathcal{R}^2(\mathcal{M}_g) = 1$. If $g \geq 3$ then we saw in the proof of 4.8.3 that the class κ_1 of e_1^A does not vanish, so $I^2(\mathcal{M}_g)$ is a proper subspace of the two-dimensional space $\mathcal{R}^2(\mathcal{M}_g)$. ☐

We therefore find that the Kawazumi–Zhang invariant is the only invariant (up to additive and multiplicative constants) that arises on $\mathcal{M}_g$ from tautological forms.

4.9 Tautological $2d$-forms

In the previous section we have given a complete description of the vector space $\mathcal{R}^2(\mathcal{C}_g^r)$ of tautological 2-forms on the space $\mathcal{C}_g^r$. We observed that for high values of r no 'new' tautological forms appear; that is: for $r > 2$ the space $\mathcal{R}^2(\mathcal{C}_g^r)$ is spanned by pullbacks of 2-forms in $\mathcal{R}^2(\mathcal{C}_g^2)$ along tautological submersions.

This observation generalizes to higher degrees, too.

Theorem 4.9.1. Let $d \geq 0$ be an integer. For all $r > 2d$ the space $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ is spanned by pullbacks of tautological $2d$-forms on $\mathcal{C}_g^{2d}$ along tautological submersions $\mathcal{C}_g^r \to \mathcal{C}_g^{2d}$.

Proof. Let $r > 2d$ be given. By Theorem 4.6.4, it holds that the vector space $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ is spanned by tautological forms associated to contracted r-marked graphs Γ of characteristic $r - d$. Let Γ be such a graph. Lemma 3.8.3 implies that the number of marked vertices of positive degree is at most $2d$. Let $\phi : \{1, \ldots, 2d\} \to \{1, \ldots, r\}$ be an injective map, such that every $i \in \{1, \ldots, r\}$ with $\deg(m(i)) > 0$ lies in the image of ϕ. By Lemma 3.8.5 it follows that Γ is in the image of the pushforward map

$$\phi_* : \mathrm{CG}(2d, 2d - d) \to \mathrm{CG}(r, r - d),$$

and Proposition 4.5.3 thus implies that α_Γ lies in the image of the pullback map

$$f^{\phi,*} : \mathcal{R}^{2d}(\mathcal{C}_g^{2d}) \to \mathcal{R}^{2d}(\mathcal{C}_g^r).$$

☐

In Theorem 4.8.2 we gave a closed formula for the dimension of $\mathcal{R}^2(\mathcal{C}_g^r)$ in terms of r and g. In particular, we saw that the growth rate of $\dim \mathcal{R}^2(\mathcal{C}_g^r)$ as r tends to infinity is quadratic. This latter statement can be generalized to arbitrary degree.

Theorem 4.9.2. Let $d \geq 0$. There exists a polynomial f_d of degree $2d$ (that does not depend on g), whose leading coefficient equals $1/(2^d \cdot d!)$, such that

$$\dim \mathcal{R}^{2d}(\mathcal{C}_g^r) \leq f_d(r) \quad \text{for all } r \geq 0, g \geq 2.$$

Proof. By Theorem 4.6.4 $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ is spanned by forms associated to contracted r-marked graphs of characteristic $r - d$. Using Theorem 3.8.1 we find that the number of such graphs in terms of r is given by a degree $2d$ polynomial f_d with leading coefficient equal to $1/(2^d \cdot d!)$. ☐

In the case $d = 1$, we have seen in the last section that at most one relation appears among the tautological forms associated to graphs that span $\mathcal{R}^2(\mathcal{C}_g^r)$, and that this relation can be obtained from $\mathcal{R}^2(\mathcal{M}_g)$ via pullback. In other words: the linear relations among tautological forms associated to graphs in $\mathcal{R}^2(\mathcal{C}_g^r)$ for low values of r determine the linear relations among these forms in $\mathcal{R}^2(\mathcal{C}_g^r)$ for general r. This allows us to prove that the dimension of $\mathcal{R}^2(\mathcal{C}_g^r)$ is given by a quadratic polynomial. This polynomial does depend on g, but stabilizes for $g > 2$.

It seems natural that this result would generalize as follows. For any $d \geq 1$, any linear relations among forms associated to graphs in $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ for high r (say, $r > 2d$) would be obtained by pulling back such relations from $\mathcal{R}^{2d}(\mathcal{C}_g^{2d})$. Then, by combining arguments from sections 4.8 and 3.8, one might be able to prove that the dimension of $\mathcal{R}^{2d}(\mathcal{C}_g^r)$ is given by a polynomial of degree $2d$. The polynomial would depend on g, but might stabilize for high values of g. One of the main problems the author encounters is that the inclusion-exclusion principle, that aids us in proving that the number of r-marked graphs of a certain characteristic is given by a polynomial, does not translate well into the language of vector spaces we use in this section: while taking the intersection of sets is distributive over taking unions, the same cannot be said about taking intersections of vector subspaces and spans of vector subspaces.

4.10 Relations induced by Abel–Jacobi maps

In [Ran12] Randal-Williams constructs cohomology classes $\Omega_A \in H^2(\mathcal{C}_g^r; \mathbb{Z})$ whose $(g + 1)$st power is torsion and hence trivial when passed to cohomology with rational coefficients. These cohomology classes are tautological and can therefore be expressed as linear combinations of the 'standard' tautological classes Δ_{ij}, K_i, and κ_i. Taking the $(g+1)$st power, then, yields relations between these tautological classes. Moreover, fiber integrating these relations then gives relations between tautological classes on $\mathcal{M}_g$.

In this section we will take a similar approach to obtain relations between tautological differential forms. Recall from Sections 1.4 and 2.7 that on the universal

Jacobian we have a canonical line bundle $\mathcal{B}$, equipped with a canonical admissible metric. We denote by $2\omega_0$ the first Chern form of this hermitian line bundle. In this section we will construct morphisms $\mathcal{C}_g^r \to \mathcal{J}_g$ and show that the pullbacks of $2\omega_0$ along these morphisms are tautological differential forms. Moreover, we will see that the $(g+1)$st power of ω_0 vanishes, and we will use this to generate relations among tautological forms.

Let $f : \mathcal{C} \to S$ be a family of curves of genus $g \geq 2$, and let $\mathcal{J} \to S$ denote the associated Jacobian family. Let $r \geq 0$ be any integer, and let $m = (m_1, \ldots, m_r)$ be an r-tuple of integers whose sum equals zero. Consider the submersion

$$p = p_{(r)} : \mathcal{C}^{r+1} \to \mathcal{C}^r : (x_1, \ldots, x_{r+1}) \mapsto (x_1, \ldots, x_r)$$

and its r sections

$$\sigma_i : \mathcal{C}^r \to \mathcal{C}^{r+1} : (x_1, \ldots, x_r) \to (x_1, \ldots, x_r, x_i) \quad (1 \leq i \leq r)$$

Now consider the following line bundle on $\mathcal{C}^{r+1}$:

$$L_m = O(m_1\sigma_1 + \cdots + m_r\sigma_r) = O(\sigma_1)^{\otimes m_1} \otimes \cdots \otimes O(\sigma_r)^{\otimes m_r}$$

The restriction of this line bundle to each fiber of p has degree 0, and this line bundle hence determines a section of the Jacobian family $\mathcal{J} \times_S \mathcal{C}^r \to \mathcal{C}^r$ associated to p. The composition of this section with the projection $\mathcal{J} \times_S \mathcal{C}^r \to \mathcal{J}$ is the morphism

$$f_m : \mathcal{C}^r \to \mathcal{J} : ((x_1, \ldots, x_r) \in \mathcal{C}_s^r) \mapsto ([O(m_1x_1 + \cdots + m_rx_r)] \in \mathcal{J}_s = \mathrm{Jac}(\mathcal{C}_s)).$$

We obtain from Proposition 1.4.14 a canonical isometry

$$f_m^*\mathcal{B}^{\otimes -1} \xrightarrow{\sim} \langle L_m, L_m \rangle_p.$$

As the Deligne pairing is bilinear, we find another canonical isometry

$$\langle L_m, L_m \rangle_p \xrightarrow{\sim} \bigotimes_{i=1}^{r} \bigotimes_{j=1}^{r} \langle O(\sigma_i), O(\sigma_j) \rangle^{\otimes m_i m_j}.$$

Notice that for all $1 \leq j \leq r$ we have a canonical isometry

$$O(\sigma_j) \xrightarrow{\sim} p_{j,r+1}^* O(\Delta)$$

so taking the pullback along σ_i yields yet another canonical isometry

$$\sigma_i^* O(\sigma_j) \simeq \sigma_i^* p_{j,r+1}^* O(\Delta) \simeq \begin{cases} p_{ji}^* O(\Delta) = p_{ij}^* O(\Delta) & \text{if } i \neq j \\ p_j^* \Delta^* O(\Delta) = p_j^* \omega^{\otimes -1} & \text{if } i = j. \end{cases}$$

By combining the above canonical isometries we obtain

$$f_m^* \mathcal{B}^{\otimes -1} \simeq \bigotimes_{1 \leq i < j \leq r} p_{ij}^* O(\Delta)^{\otimes 2 m_i m_j} \otimes \bigotimes_{i=1}^{r} p_i^* \omega^{\otimes -m_i^2}.$$

Universally we obtain the following result.

Proposition 4.10.1. Let $r \geq 0$ be an integer, and let $(m_1, \ldots, m_r)$ be a tuple of integers whose sum equals zero. Consider the morphism of stacks

$$f_m : \mathcal{C}_g^r \to \mathcal{J}_g$$

that takes a family $f : \mathcal{C} \to S$ with sections $\sigma_1, \ldots, \sigma_r$ and maps it to the pair (f, σ), with σ the following section of the Jacobian family $\mathcal{J}_f \to S$:

$$\sigma : S \to \mathcal{J}_f : s \mapsto [O(m_1\sigma_1(s) + \cdots + m_r\sigma_r(s))] \in \mathrm{Jac}(\mathcal{C}_s).$$

Then we have a canonical isometry of line bundles on $\mathcal{C}_g^r$:

$$f_m^*\mathcal{B}^{\otimes -1} \simeq \bigotimes_{1\leq i<j\leq r} p_{ij}^* O(\Delta)^{\otimes 2m_im_j} \otimes \bigotimes_{i=1}^{r} p_i^*\omega^{\otimes -m_i^2}. \qquad \square$$

Taking first Chern forms then yields:

Corollary 4.10.2. Let $r \geq 0$ be an integer, and let $m = (m_1, \ldots, m_r)$ be a tuple of integers whose sum equals zero. Consider the induced morphism of stacks $f_m : \mathcal{C}_g^r \to \mathcal{J}_g$ as described in Proposition 4.10.1. Then we have the following equality of 2-forms on $\mathcal{C}_g^r$:

$$-2f_m^*\omega_0 = \sum_{1\leq i<j\leq r} 2m_im_jp_{ij}^*h + \sum_{i=1}^{r} m_i^2p_i^*e^A \in A^2(\mathcal{C}_g^r).$$

In particular the form $f_m^*\omega_0$ is tautological. $\square$

Example 4.10.3. Set $r = 2$ and $m = (-1, 1)$, then the associated morphism

$$f_{(-1,1)} : \mathcal{C}_g^2 \to \mathcal{J}_g$$

equals the Abel–Jacobi morphism δ defined in Section 1.4 and Section 2.7. From Proposition 4.10.1 we retrieve the canonical isometry

$$\delta^*\mathcal{B}^{\otimes -1} \xrightarrow{\sim} O(\Delta)^{\otimes -2} \otimes p_1^*\omega^{\otimes -1} \otimes p_2^*\omega^{\otimes -1}$$

that we already encountered in Proposition 1.4.15. Taking Chern forms, then, yields the identity

$$-2\delta^*\omega_0 = -2h + p_1^*e^A + p_2^*e^A$$

that was proved in Proposition 4.3.6 and [dJon16, Theorem 1.4].

The following proposition will be used to obtain relations among tautological differential forms.

Proposition 4.10.4 ([dJon20, Proposition 5.1])**.** Let $2\omega_0 \in A^2(\mathcal{J}_g)$ be the first Chern form of the canonical line bundle $\mathcal{B}$ on $\mathcal{J}_g$ with its canonical admissible metric. Then we have

$$\omega_0^{g+1} = 0 \in A^{2g+2}(\mathcal{J}_g). \qquad \square$$

For instance, the $(g+1)$st power of the form induced by the Abel–Jacobi map in Example 4.10.3 is zero, and we can express this $(g+1)$st power in terms of tautological forms. In this example we will compute the resulting relation in the case $g = 2$, as this is still feasible to do by hand.

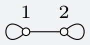
Example 4.10.5. Suppose that $g = 2$ and $r = 2$. In this case, we obtain from Example 4.10.3 and Corollary 4.10.4:

$$(-2h + p_1^* e^A + p_2^* e^A)^3 = 0 \in \mathcal{R}^6(\mathcal{C}_2^2).$$

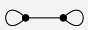
By expanding parentheses we obtain a linear combination of 10 tautological forms on $\mathcal{C}_g^2$ associated to 2-marked graphs. Of these tautological forms we can take the fiber integral along the map $\mathcal{C}_g^2 \to \mathcal{M}_g$. For instance, consider the form

$$\alpha = h \wedge p_1^* e^A \wedge p_2^* e^A \in \mathcal{R}^6(\mathcal{C}_2^2).$$

This form is the tautological form associated to the 2-marked graph

$\Gamma = \Bigg($  $\Bigg).$

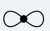
The projection $\mathcal{C}_g^2 \to \mathcal{M}_g$ is the tautological morphism associated to the map $\emptyset \to \{1, 2\}$. Therefore, by Proposition 4.5.4, the fiber integral of α along this projection is the tautological form associated to the graph

$\phi^*\Gamma = \Bigg($  $\Bigg)$

We can compute the tautological form associated to $\phi^*\Gamma$ by contracting this graph: we have:

$$\alpha_{\phi^*\Gamma} = \alpha_{\tilde{\varrho}_2(\phi^*\Gamma)} = \alpha_{\varrho(\phi^*\Gamma)},$$

and $\varrho(\phi^*\Gamma)$ is the contracted graph

$\varrho(\phi^*\Gamma) = \Bigg($  $\Bigg)$

The tautological form associated to this graph is e_1^A, and we find:

$$\int_{\mathcal{C}_2^2 \to \mathcal{M}_2} h \wedge p_1^* e^A \wedge p_2^* e^A = e_1^A.$$

By repeating this procedure for all the 10 tautological forms we found earlier, we obtain the following identity:

$$-8\nu - 12 e_1^A = 0 \in A^2(\mathcal{M}_g).$$

The identity we obtain in Example 4.10.5 can be derived directly from [dJon16]. Proposition 9.1 of loc. cit. gives an identity of 2-forms on $\mathcal{M}_g$

$$e_1^J - e_1^A = \frac{2g-2}{2g+1} \cdot \frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}},$$

where e_1^J is a 2-form on $\mathcal{M}_g$ that vanishes on the locus of hyperelliptic curves in $\mathcal{M}_g$ (by loc. cit., Proposition 10.7). Every curve of genus 2 is hyperelliptic, so for $g = 2$ we obtain the following relation:

$$-e_1^A = \frac{2}{5} \cdot \frac{\partial\overline{\partial}\varphi}{\pi\sqrt{-1}} = \frac{2}{5} \cdot (\nu - e_1^A),$$

from which one easily derives the identity found in Example 4.10.5.

4.11 Computations in higher degrees and genera

In Example 4.10.5 we used Corollary 4.10.2 and Corollary 4.10.4 to obtain a relation among tautological forms in $\mathcal{R}^2(\mathcal{M}_2)$. This was relatively easy, as we only needed to work with the third power of ω, and hence needed to compute the fiber integral of 'only' 10 differential forms. Of course, if we want to construct similar relations in higher genera, or in higher degrees, it quickly becomes infeasible to do this by hand. In this section, we describe an algorithm for finding relations among generators of spaces $\mathcal{R}^{2d}(\mathcal{C}_g^s)$, and provide some example computations.

Recall that in Section 4.7 we have constructed a surjective graded homomorphism $\mathbb{R}[\mathrm{CG}(r)] \to \mathcal{R}^{2*}(\mathcal{C}_g^r)$. We will denote this morphism by α_r. Computations in the ring $\mathbb{R}[\mathrm{CG}(r)]$ can be carried out effectively. We will be using the following lemma to construct elements in the kernel of α_r.

Lemma 4.11.1. Let $r \geq 2$ be an integer. For $1 \leq i, j \leq r$ let Γ_{ij} be the r-marked graph with no unmarked vertices and a single edge between the vertices marked i and j; notice that Γ_{ij} is contracted as it has no unmarked vertices, and notice that $\Gamma_{ij} = \Gamma_{ji}$. Consider the polynomial ring

$$\mathbb{R}[\mathrm{CG}(r)][x_1, \ldots, x_{r-1}],$$

and define $x_r = -x_1 - \cdots - x_{r-1}$. Now define the polynomial

$$W_r = \sum_{i,j=1}^{r} \Gamma_{ij} \cdot x_i x_j \in \mathbb{R}[\mathrm{CG}(r)][x_1, \ldots, x_{r-1}].$$

Then W_r^{g+1} lies in the kernel of the homomorphism

$$\bar{\alpha}_r : \mathbb{R}[\mathrm{CG}(r)][x_1, \ldots, x_{r-1}] \to \mathcal{R}^{2*}(\mathcal{C}_g^r)[x_1, \ldots, x_{r-1}]$$

induced by α_r. In particular, all coefficients of W_r^{g+1} lie in the kernel of α_r.

Proof. Set $w_r = \alpha_r(W_r)$. By Corollary 4.10.2 we then have for all $m_1, \ldots, m_{r-1} \in \mathbb{Z}$:

$$w_r(m_1, \ldots, m_{r-1}) = -2f_m^*\omega_0 \in \mathcal{R}^2(\mathcal{C}_g^r)$$

where m denotes the tuple $(m_1, \ldots, m_{r-1}, -m_1 - \cdots - m_{r-1})$. By Proposition 4.10.4 we then see that w_r^{g+1} vanishes on $\mathbb{Z}^{r-1}$, which implies that it must be the zero polynomial. ☐

Let $s \leq r$ be an integer, and consider the inclusion map $\phi : \{1, \ldots, s\} \to \{1, \ldots, r\}$. From Diagram 4.7.2 and Proposition 4.5.4 we obtain a commutative diagram

$$\begin{tikzcd}
\mathbb{R}[\mathrm{CG}(r)] \arrow[r, "\alpha_r"] \arrow[d, "\phi^*"'] & \mathcal{R}^{2*}(\mathcal{C}_g^r) \arrow[d, "\int_{f\phi}"] \\
\mathbb{R}[\mathrm{G}(s)] \arrow[r] \arrow[d, "\tilde{\varrho}_g"'] & \mathcal{R}^{2*}(\mathcal{C}_g^s) \\
\mathbb{R}[\mathrm{CG}(s)] \arrow[ur, "\alpha_s"'] &
\end{tikzcd} \tag{4.11.2}$$

In particular, we may pass the coefficients of W_r^{g+1} through the homomorphism $\tilde{\varrho}_g \circ \phi^*$ to obtain elements in the kernel of α_s.

Example 4.11.3. Let $r = 2, s = 0, g = 2$. We have:

$$W_r = \Gamma_{11}x_1^2 + \Gamma_{12}x_1x_2 + \Gamma_{21}x_2x_1 + \Gamma_{22}x_2^2 = (\Gamma_{11} - 2\Gamma_{12} + \Gamma_{22})x_1^2,$$

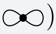
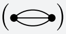
and

$$W_r^3 = (\Gamma_{11} - 2\Gamma_{12} + \Gamma_{22})^3x_1^6.$$

From Lemma 4.11.1 we find that $(\Gamma_{11} - 2\Gamma_{12} + \Gamma_{22})^3$ lies in the kernel of α_r. In other words: we have the following identity of tautological forms on $\mathcal{C}_2^2$:

$$(p_1^*e^A - 2h + p_2^*e^A)^3 = 0,$$

which was already clear from Proposition 4.3.6 and Proposition 4.10.4. A computation by hand shows:

$(\tilde{\varrho}_g \circ \phi_*)((\Gamma_{11} - 2\Gamma_{12} + \Gamma_{22})^3) = -12\,($  $) - 8\,($  $) \in \mathrm{Ker}(\alpha_s)$,

and applying α_s to this element of $\mathbb{R}[\mathrm{CG}(s)]$ then yields the identity

$$-12e_1^A - 8\nu = 0 \in \mathcal{R}^2(\mathcal{M}_2)$$

we found in Example 4.10.5.

Another trick we can use is the following. Instead of viewing the genus g as a constant, we view it as a variable. In Diagram 4.11.2 we replace the base ring $\mathbb{R}$

in the left column by the polynomial ring $\mathbb{R}[g']$. We thus obtain for all $g \geq 2$ a commutative diagram

$$\begin{tikzcd}
\mathbb{R}[g'][\mathrm{CG}(r)] \arrow[r, "\alpha_r"] \arrow[d, "\phi^*"'] & \mathcal{R}^{2*}(\mathcal{C}_g^r) \arrow[d, "\int_{f\phi}"] \\
\mathbb{R}[g'][\mathrm{G}(s)] \arrow[r] \arrow[d, "\tilde{\varrho}"'] & \mathcal{R}^{2*}(\mathcal{C}_g^s) \\
\mathbb{R}[g'][\mathrm{CG}(s)] \arrow[ur, "\alpha_s"'] &
\end{tikzcd} \tag{4.11.4}$$

Here $\tilde{\varrho}$ denotes the unique morphism of $\mathbb{R}[g']$-algebras that maps a graph $\Gamma \in \mathrm{G}(s)$ to $\lambda_\Gamma \cdot \varrho(\Gamma)$, where λ_Γ is given by

$$\lambda_\Gamma = 0^a (2 - 2g')^b \in \mathbb{R}[g']$$

with a and b the number of connected components of Γ with no marked vertices of characteristic 1 and 0, respectively. The maps from left to right in diagram 4.11.4 are the unique extensions of the corresponding maps in diagram 4.11.2 that map g' to g.

Our algorithm for finding elements in the kernel of α_s is as follows. We pick $r \geq 2$, $0 \leq s \leq r$, and $G \geq 2$. We then compute $W_r^{G+1} \in \mathbb{R}[\mathrm{CG}(r)][x_1, \ldots, x_{r-1}]$. Then for each of the coefficients $c \in \mathbb{R}[\mathrm{CG}(r)]$ of W_r^{G+1}, compute $\tilde{\varrho}(\phi^* c)$. The resulting element is in the kernel of the homomorphism $\mathbb{R}[g'][\mathrm{CG}(s)] \to \mathcal{R}^{2*}(\mathcal{C}_g^s)$ for all $2 \leq g \leq G$.

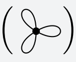
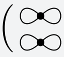
An implementation of this algorithm in Sage is provided in [vdLug21]. We will list some results for low values of r, s, G.

Example 4.11.5. If we set $r = 2$, $s = 0$, and $G = 4$, we find that for all $2 \leq g \leq 4$, the following element lies in the kernel of the map $\mathbb{R}[\mathrm{CG}(0)] \to \mathcal{R}^2(\mathcal{M}_g)$:

$-3(g-4)(g-3)(g-1)$ () $- 2(g-4)(g-3)$ () .

If $g = 2$ we retrieve Example 4.11.3. For $g = 3$ and $g = 4$ we retrieve nothing at all, as the above vector vanishes.

Example 4.11.6. If we set $r = 2$, $s = 0$ and $G = 3$, we obtain the following element in the kernel of $\mathbb{R}[\mathrm{CG}(0)] \to \mathcal{R}^{2*}(\mathcal{M}_g)$ for all $g \in \{2, 3\}$:

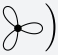
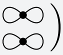
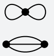
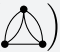
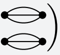
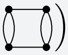
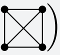
$-8g$ () $+ 3$ () $- 32$ () $+ 24$ () $+ 8$ ()

In terms of differential forms we obtain the following relation among tautological forms in $\mathcal{R}^4(\mathcal{M}_g)$ for $g \in \{2, 3\}$:

$$-8g \cdot e_2^A + 3(e_1^A)^2 - 32 \int (p_1^* e^A \wedge h^3) + 24 \int (p_1^* e^A \wedge p_2^* e^A \wedge h^2) + 8 \int h^4 = 0$$

where the integral symbol denotes fiber integration along the map $\mathcal{C}_g^2 \to \mathcal{M}_g$.

Example 4.11.7. If we set $r = 4$, $s = 0$ and $G = 5$, we obtain another element in $\mathbb{R}[\mathrm{CG}(0)]$ that involves all the 11 contracted unmarked graphs of characteristic -2. For $2 \leq g \leq 5$, the following element lies in the kernel of the map $\mathbb{R}[\mathrm{CG}(0)] \to \mathcal{R}^4(\mathcal{M}_g)$.

$-8(g-4)(4g^2-20g+3)\Big($  $\Big) + (9g^2-87g+201)\Big($  $\Big)$
$-32(g-5)(4g-15)\,($  $) + (72g^2-696g+1608)\,($  $)$
$+24(g-5)(g-4)\,($  $) - 4(g-2)\Big($  $\Big) - 48(g-2)\Big($  $\Big)$
$-48(g-5)\Big($  $\Big) - 4\Big($  $\Big) - 72\Big($  $\Big) - 48\Big($  $\Big)$.

Our algorithm takes integers r, s, G and gives relations in $\mathcal{R}^{2d}(\mathcal{C}_g^s)$, where $d = G + 1 - r + s$. It is interesting to observe what happens to these relations when we fix s and d and let G (and hence $r = G + 1 + s - d$) increase.

For example, fixing $s = 0$, $d = G + 1 - r + s = 1$ and running our algorithm with G increasing from 2 to 5 yields the following elements of $\mathbb{R}[g][\mathrm{CG}(0)]$:

G	vectors in $\mathbb{R}[g][\mathrm{CG}(0)]$
2	$(-3(g-1)\,($  $) - 2\,($  $))$
3	$(g-3)\,(-3(g-1)\,($  $) - 2\,($  $))$
4	$(g-4)(g-3)\,(-3(g-1)\,($  $) - 2\,($  $))$
5	$(g-5)(g-4)(g-3)\,(-3(g-1)\,($  $) - 2\,($  $))$

The pattern is clear: it seems that for $G \geq 2$ the following vector is obtained in $\mathbb{R}[g][\mathrm{CG}(0, -1)]$:

$\left(\prod_{k=3}^{G}(g-k)\right) \cdot (-3(g-1)\,($  $) - 2\,($  $))$.

In particular, the only value of g for which this vector yields a nontrivial relation in the ring of tautological forms would then be $g = 2$.

A similar phenomenon occurs if we increase G in examples 4.11.6 and 4.11.7. This suggests that relations (or at least, relations found using ω_0) among elements of $\mathcal{R}^*(\mathcal{M}_g)$ for low values of g vanish if we let g increase. These observations prompt the following question.

Question 4.11.8. Suppose we are given integers $r \geq 0$ and $d \geq 0$. Does there exist a $g_0 \geq 2$ such that for all $g \geq g_0$ the linear map

$$\mathbb{R}[\mathrm{CG}(r, r-d)] \to \mathcal{R}^{2d}(\mathcal{C}_g^r)$$

is an isomorphism? Is there an expression for g_0 in terms of r and d?

Recall that an analogue in rings of tautological classes is given by Mumford's conjecture (proved in [MW07]) that for any $d > 0$ there exists a $g_0 \geq 2$ such that the map $\mathbb{Q}[\kappa_1, \kappa_2, \dots] \to RH^*(\mathcal{M}_g)$ is an isomorphism in degree d for all $g \geq g_0$.

Theorem 4.8.2 moreover states that the above question can be answered with 'yes' if $d = 1$. In this case, we have:

$$\dim \mathbb{R}[\mathrm{CG}(r, r-1)] = \tfrac{1}{2}r^2 + \tfrac{1}{2}r + 2$$

by Example 3.7.2, and for all $g \geq g_0 = 3$ we have:

$$\dim \mathcal{R}^2(\mathcal{C}_g^r) = \tfrac{1}{2}r^2 + \tfrac{1}{2}r + 2$$

by Theorem 4.8.2, and the linear map

$$\mathbb{R}[\mathrm{CG}(r, r-d)] \to \mathcal{R}(\mathcal{C}_g^r)$$

is therefore an isomorphism, as it is a surjective map between vector spaces of the same dimension. It is moreover interesting to see that in this case the value $g_0 = 3$ does not depend on r.

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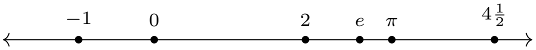
Summary

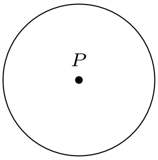
The study of moduli spaces could be viewed as the mathematical analogue of taxonomy in biology. Where a biologist would, for example, try to find all the distinct species of Galapagos finches, a mathematician would be interested in finding all the mathematical objects of a certain type. When making such classifications, one can make use of moduli spaces: geometrical objects whose points correspond one-to-one with the objects we wish to classify.

For instance, take all the numbers and arrange them in ascending order to obtain a line, the *number line*:

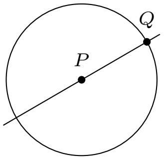


The points on the number line correspond one-to-one with the numbers. Therefore, the number line is a *moduli space* for all numbers.

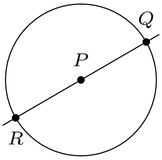
Now, let us classify all lines in the plane that go through a certain point. Suppose that P is a point in the plane, and suppose that we wish to classify all lines in the plane through P. We will therefore look for a moduli space for lines through P. Now draw a circle around P that has P as its center:



If Q is a point on this circle, then we obtain a line through P by drawing a line through P and Q:

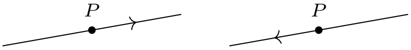


Moreover, every line through P can be obtained in this way. It appears that we have found a moduli space for all lines through P. This circle, however, is *not* a moduli space. Indeed, if it were a moduli space, its points would correspond one-to-one with lines through P. But consider two points Q and R that lie on opposite sides of the circle:

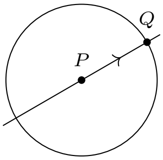


The line through P and Q and the line through P and R are the same! The correspondence between points on the circle and lines through P is 'two-to-one', instead of one-to-one: we have pairs of points that induce the same line.

So how do we get a moduli space of lines through P? One thing we can do is to add extra structure to the lines we are trying to classify. For example, we could rather try looking at lines through P with a *direction.* Every line in the plane has two directions:



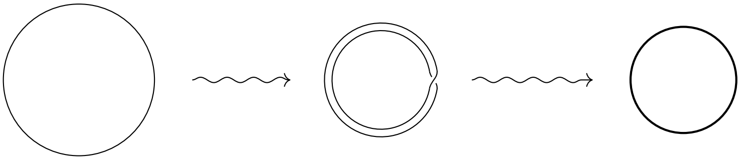
If we now have a point Q on the circle around P, then we draw the line through P and Q in the direction from P to Q:



The reader can verify that even though two points at opposite sides of the circle give rise to the same lines, the directions of these lines differ. Points on the circle therefore correspond one-to-one with lines through P with a direction. We thus find that the circle around P is a moduli space for lines through P with a direction.

This method of adding extra structure to objects so that their moduli space will be easier to describe is called *rigidification.* It is a technique that is often applied by mathematicians when they wish to study moduli spaces of more complicated objects.

But what if we do not want to add any additional structure to the objects we are classifying? If we are only interested in lines through P, and not at all in lines through P with a direction, then we can use a different technique. Manipulate the circle as follows: lift the circle from the plane, wrap it 'around itself' once, and glue the two strands together:



We thus obtain a new circle. Any two points on opposite sides of the old circle are glued together into a single point on the new circle. In other words: every point on the new circle corresponds to a pair of points on opposite sides of the old circle, and these points induce the same line through P. So there is a one-to-one correspondence between points on the new circle and lines through P, and the new circle is therefore a moduli space for the lines through P.

We invite the reader to carry out the construction described above. Take a sheet of paper, a couple of markers with different colors, and a rubber band. Draw the point P on the sheet of paper, and lay down the rubber band around it such that it forms a circle around this point. For each marker choose a pair of points on opposite sides of the rubber band, and mark these points with the same color. Now take the rubber band and wrap it around itself once, as described in the above figure. You will see that all pairs of points with the same color coincide.

'Wrapping' a geometric object around itself to obtain a new geometric object is also a technique mathematicians use often. They call this technique *taking a quotient*.

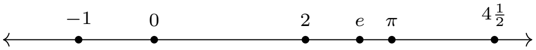
In this thesis we study the moduli space of certain geometric objects, namely compact Riemann surfaces of genus g. Moreover we discuss the differential forms that live on this moduli space. In Chapter 1 we discuss some theory about submersions of manifolds, families of compact Riemann surfaces, and hermitian line bundles on these families. We construct various canonical hermitian line bundles and give canonical isometries between these line bundles. In Chapter 2 we look at the moduli space of compact Riemann surfaces of genus g. We discuss that there is no 'nice' moduli space, but that this problem can be fixed by rigidifying or by taking quotients. In Chapter 3 we look at *marked graphs.* These are graphs of which some vertices are marked with positive integers. These graphs can be contracted, and we show that formulas can be given for the number of contracted marked graphs of any given characteristic in terms of the number of marked vertices. Finally, in Chapter 4, we study *tautological differential forms* on the moduli space. By using the marked graphs from Chapter 3 we can show that there are not 'too many' such tautological differential forms, and we compute some relations between these tautological differential forms.

Samenvatting

Tautologische differentiaalvormen op moduliruimtes van krommen

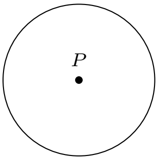
Het bestuderen van moduliruimtes zou opgevat kunnen worden als de wiskundige variant van de taxonomie uit de biologie. Waar een bioloog bijvoorbeeld zou proberen om alle verschillende soorten vinken op de Galapagoseilanden te vinden, zo zou een wiskundige geïnteresseerd kunnen zijn in het vinden van alle wiskundige objecten van een bepaald type. Bij zo'n classificatie kan men gebruik maken van moduliruimtes: meetkundige objecten waarvan de punten één-op-één overeenkomen met de objecten die we willen classificeren.

Bijvoorbeeld: door alle getallen op volgorde achter elkaar te leggen, verkrijgen we een lijn, de *getallenlijn*:

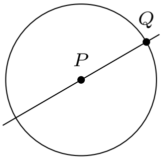


De punten op de getallenlijn komen één-op-één overeen met de getallen. Daarom is de getallenlijn een *moduliruimte* van alle getallen.

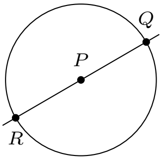
Laten we nu proberen om alle lijnen in het vlak die door een bepaald punt gaan te classificeren. Laat P een punt in het vlak zijn, en stel dus dat we alle lijnen in het vlak door P willen classificeren. We gaan dus op zoek naar een moduliruimte voor lijnen door P. Teken nu een cirkel rond P die P als middelpunt heeft:



Als nu Q een punt op deze cirkel is, dan krijgen we een lijn door P door een lijn door P en Q te trekken:

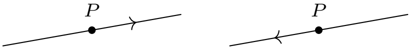


Andersom kan iedere lijn door P op deze manier worden verkregen. We lijken dus een moduliruimte te hebben gevonden voor alle lijnen door P. Toch is deze cirkel géén moduliruimte. Immers, als deze cirkel wél een moduliruimte zou zijn, dan zouden de punten op deze cirkel één-op-één overeen moeten komen met lijnen door P. Maar neem twee punten P en Q die aan weerszijden van de cirkel liggen:

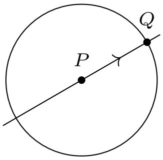


De lijn door P en Q en de lijn door P en R zijn hetzelfde! De overeenkomst tussen punten op de cirkel en lijnen door P is dus 'twee-op-één', in plaats van één-op-één: er zijn steeds tweetallen punten die dezelfde lijn geven.

Hoe krijgen we nu een moduliruimte van alle lijnen door P? Eén manier is het toekennen van extra structuur aan de lijnen die we willen classificeren. We kunnen bijvoorbeeld proberen te kijken naar lijnen door P met een *richting*. Iedere lijn in het vlak heeft twee richtingen:



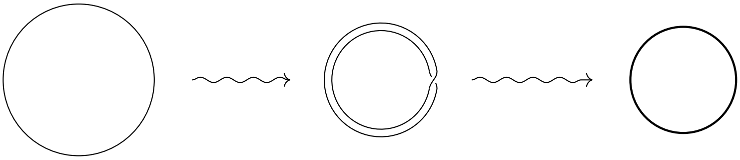
Als we nu een punt Q op de cirkel rond P hebben, dan tekenen we de lijn door P en Q in de richting van P naar Q:



De lezer kan nu nagaan dat twee punten aan weerszijden van de cirkel weliswaar twee dezelfde lijnen opleveren, maar dat de richtingen van deze lijnen verschillen. Punten op de cirkel komen dus één-op-één overeen met lijnen door P met een richting. We vinden dus dat de cirkel rond P een moduliruimte is voor de lijnen door P met een richting.

Deze manier, dus het toekennen van extra structuur aan objecten zodat hun moduliruimte eenvoudiger te beschrijven is, heet *rigidificatie*. Dit is een techniek die wiskundigen vaak gebruiken als ze moduliruimtes van ingewikkelde objecten willen bestuderen.

Maar wat als we geen extra structuur willen toekennen aan de objecten die we aan het classificeren zijn? Als we alleen geïnteresseerd zijn in lijnen door P, en helemaal niet in lijnen door P met een richting, dan kunnen we ook een andere techniek gebruiken. Manipuleer de cirkel als volgt: pak de cirkel van het vlak, draai de cirkel vervolgens een keer 'om zichzelf' heen, en plak de twee 'takken' nu aan elkaar vast:



We krijgen zo een nieuwe cirkel. Ieder tweetal tegenover elkaar liggende punten uit de oude cirkel worden in de nieuwe cirkel samengeplakt tot één punt. Oftewel: ieder punt op de nieuwe cirkel correspondeert met twee punten op de oude cirkel die tegenover elkaar liggen, en deze twee punten geven dezelfde lijn door P. Er is dus een één-op-één overeenkomst tussen punten op de nieuwe cirkel en lijnen door P, en de nieuwe cirkel is dus een moduliruimte voor de lijnen door P.

We nodigen de lezer uit om bovenstaande constructie zelf uit te voeren. Neem een vel papier, een paar stiften in verschillende kleuren, en een elastiek. Teken op het vel papier een punt, en leg het elastiek hier als een cirkel omheen. Kies nu voor iedere stift een tweetal punten aan weerszijden van het elastiek, en markeer deze punten met dezelfde kleur. Pak vervolgens het elastiek, en draai het elastiek een keer om zichzelf heen zoals in bovenstaande afbeelding. U zult zien dat alle tweetallen punten met dezelfde kleur nu op elkaar worden gelegd.

Het 'oprollen' van een meetkundig object om een nieuw meetkundig object te verkrijgen is ook een techniek die veel wordt toegepast door wiskundigen. Ze noemen deze techniek het nemen van een *quotiënt*.

In dit proefschrift bestuderen we de moduliruimte van bepaalde meetkundige objecten, namelijk compacte Riemannoppervlakken van geslacht g. Bovendien bestuderen we de differentiaalvormen die op deze moduliruimte leven. In Hoofdstuk 1 bespreken we theorie over submersies van variëteiten, families van compacte Riemannoppervlakken, en hermitische lijnbundels op deze families. We construeren hier enkele canonieke hermitische lijnbundels en geven canonieke isometrieën tussen deze lijnbundels. In Hoofdstuk 2 bekijken we de moduliruimte van compacte Riemannoppervlakken van geslacht g. We bespreken dat er geen 'mooie' moduliruimte bestaat, maar dat dit probleem te verhelpen is door te rigidificeren, of door quotiënten te nemen. In Hoofdstuk 3 bekijken we *gemarkeerde grafen*. Dit zijn grafen waarvan een aantal knopen is gemarkeerd met positieve gehele getallen. Deze grafen kunnen worden samengetrokken, en we laten zien dat er formules kunnen worden gegeven voor het aantal samengetrokken gemarkeerde grafen van iedere gegeven karakteristiek in termen van het aantal gemarkeerde knopen. Tot slot bestuderen we in Hoofdstuk 4 *tautologische differentiaalvormen* op de moduliruimte. Met behulp van de gemarkeerde grafen uit Hoofdstuk 3 kunnen we laten zien dat er niet 'te veel' van deze tautologische differentiaalvormen zijn, en we berekenen enkele relaties tussen deze tautologische differentiaalvormen.

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I would not have been able to write this thesis without the support of many people.

First and foremost, I would like to thank my advisor, Robin de Jong, for his support, advice, and patience throughout these years. He introduced me to the subject of this thesis, and our discussions helped me tremendously to understand how everything ties together. Moreover, he taught me how to be a better researcher, mathematician, and writer. I also look back with great pleasure to the many fun courses we taught together, and I am grateful for his confidence in my teaching abilities.

I would like to thank the people at the Mathematical Institute for a great atmosphere. Raymond, we have been friends since our first year in Leiden, and I am happy that we shared an office during our PhD programmes. I am also thankful for our discussions and your advice, both about academia and life outside it. If I ever end up running a café, it will be thanks to your lessons in entrepreneurship. Peter, thank you for pushing me to the limit to improve my aim. I also look back fondly to our football-related banter on (some) Mondays. Anna, Garnet, Julian, Misja, Rosa, Sjabbo, Wouter, thank you for occasionally dragging me away from my desk for our afternoon walks (or runs, depending on whether the ice cream van was already there). Gabriel, Steven, and all the others, I will (k)not forget the interesting, long, and hilarious discussions we had in the common room. I am also grateful to all my teammates at Mathletico for a fun and healthy distraction after a long day of work. We may not have been the greatest team ever to have graced the fields and halls of the sports center, but we certainly had the most fun.

I would also like to express my gratitude to my family. I thank my parents and brother for a warm home and their never-ending support. This extends to my grandparents; even though I cannot show them this thesis anymore, I am sure they would have been very proud.

And, finally, I am grateful to my partner, Jana, for sharing many laughs, for taking me on wonderful trips, and for inspiring and motivating me.

Curriculum vitae

Stefan van der Lugt was born on 2 July 1991 in Haarlem. In 2009 he obtained his gymnasium level diploma at the Coornhert Lyceum in Haarlem, and in the same year he started studying physics and mathematics at Universiteit Leiden. In 2012 he obtained his bachelor's degree in mathematics cum laude, and in 2015 he obtained his master's degree in mathematics cum laude.

After finishing his studies, Stefan started his PhD programme at Universiteit Leiden under the supervision of dr. Robin de Jong. He hopes to defend his PhD thesis in April 2022.